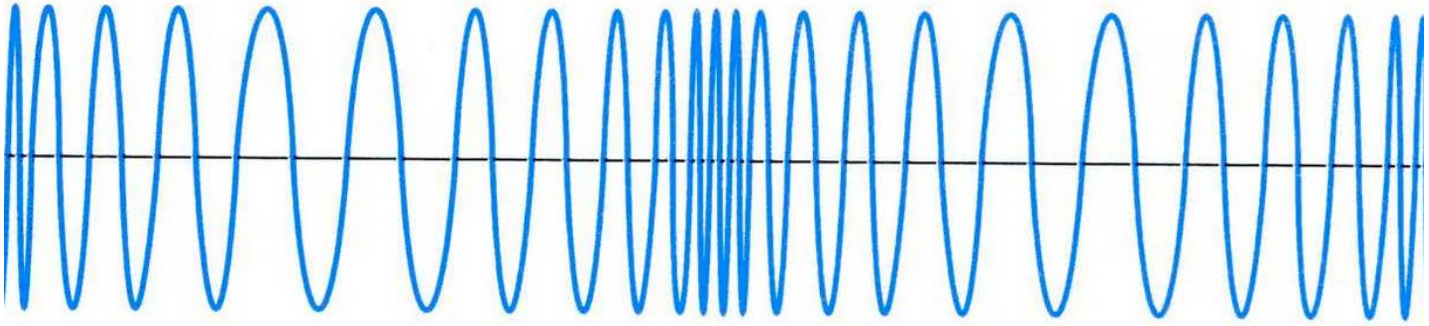


# 1 Signals

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## Objectives

Learn to:

- Perform transformations on signals.
- Use step, ramp, pulse, and exponential waveforms to model simple signals.
- Model impulse functions.
- Calculate power and energy contents of signals.



***Signals*** come in many forms: continuous, discrete, analog, digital, periodic, nonperiodic, with even or odd symmetry or no symmetry at all, and so on. Signals with special ***waveforms*** include ramps, exponentials, and impulses. This chapter introduces the vocabulary, the properties, and the ***transformations*** commonly associated with signals in preparation for exploring in future chapters ***how signals interact with systems***.

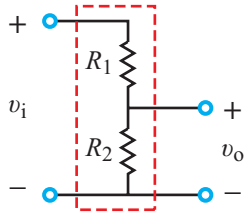



**Figure 1-1:** A voltage divider is a simple system.

## Overview

This book is about how signals interact with systems. More precisely, it is about how a system transforms input signals (***excitations***) into output signals (***responses***) to perform a certain operation (or multiple operations). A system may be as simple as the voltage divider in **Fig. 1-1**, wherein the divider scales down input voltage $v_i$ to output voltage $v_o = [R_2/(R_1 + R_2)]v_i$, or as complex as a human body (**Fig. 1-2**). Actually, the human body is a ***system of systems***; it includes the respiratory, blood circulation, and nervous systems, among many others. Each can be modeled as a system with one or more input signals and one or more output signals. When a person's fingertip touches a hot object (**Fig. 1-3**), a nerve ending in the finger senses the elevated temperature and sends a message (input signal) to the ***central nervous system*** (***CNS***), consisting of the brain and spinal cord. Upon processing the input signal, the CNS (the system) generates several output signals directed to various muscles in the person's arm, ordering them to move the finger away from the hot object.

By modeling signals and systems mathematically, we can use the system model to predict the output resulting from a specified

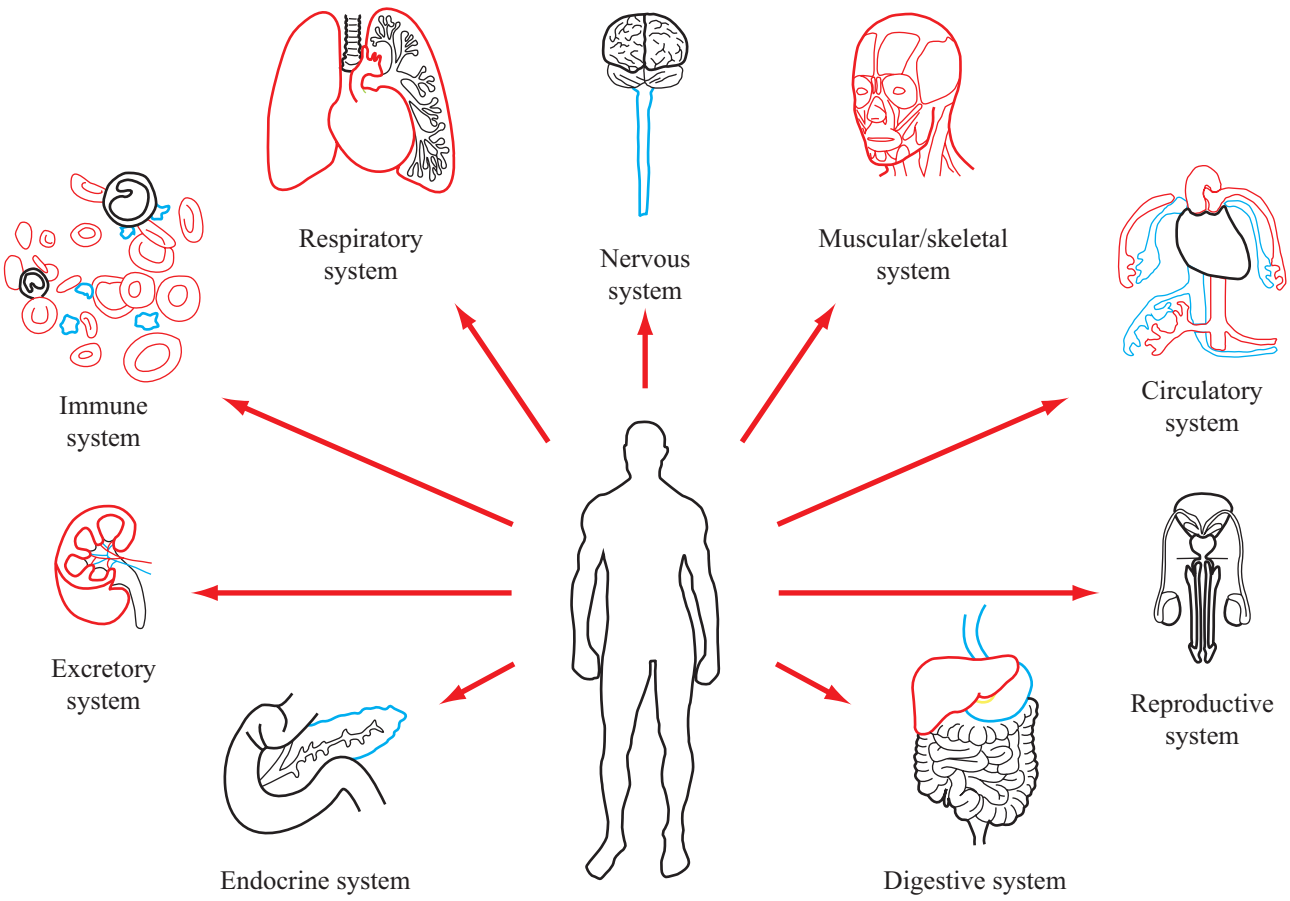



**Figure 1-2:** The human body is a system of systems.

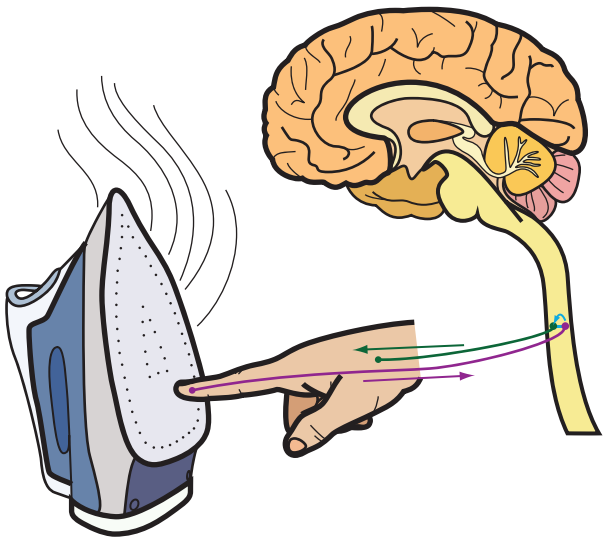


**Figure 1-3:** Finger-CNS-muscle communication.

input. We can also design systems to perform operations of interest. A few illustrative examples are depicted in **Fig. 1-4**. Signals and systems are either continuous or discrete. Both types are treated in this book, along with numerous examples of practical applications.

To set the stage for a serious study of signals and systems and how they interact with one another, we devote the current chapter to an examination of the various mathematical models and attendant properties commonly used to characterize physical signals, and then we follow suit in Chapter 2 with a similar examination for systems.

# 1-1 Types of Signals

## 1-1.1 Continuous vs. Discrete

The acoustic pressure waveform depicted in **Fig. 1-5(a)** is a ***continuous-time signal*** carrying music between a source (the trumpet) and a receiver (the listener's ear). The waveform varies with both spatial location and time. At a given instant in time, the waveform is a plot of acoustic pressure as a function of the spatial dimension $x$, but to the listener's eardrum, the intercepted waveform is a time-varying function at a fixed value of $x$.

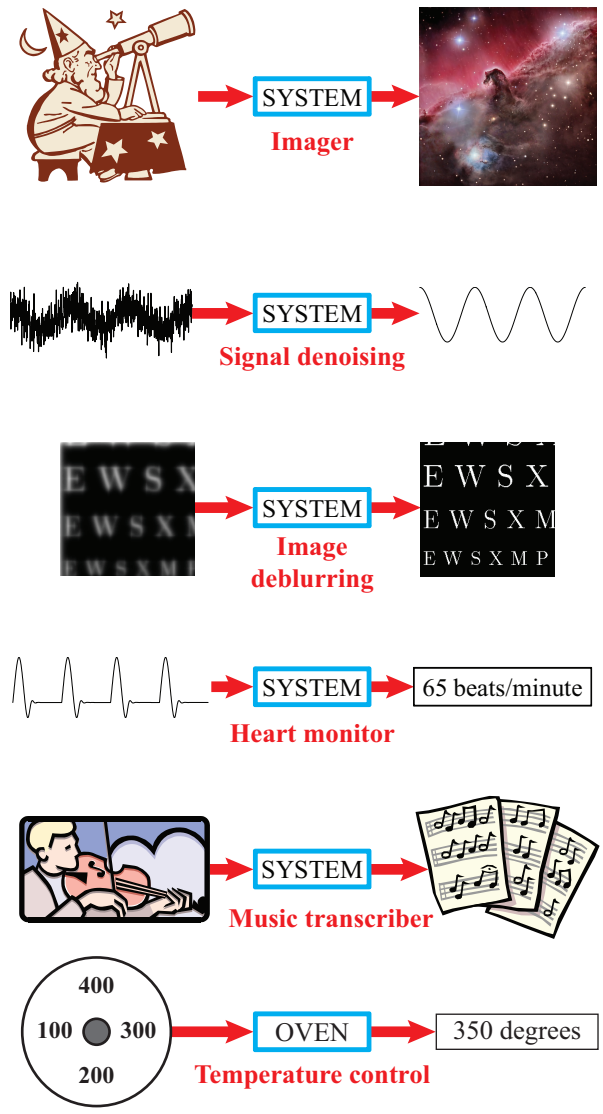


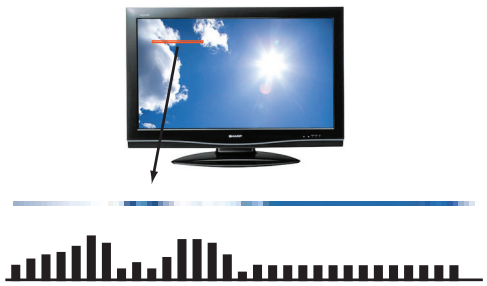

**Figure 1-4:** A system transforms a continuous input signal $x(t)$ into an output signal $y(t)$ or a discrete input signal $x[n]$ into a discrete output signal $y[n]$. Such system transformations exist not only in the few examples shown here but also in countless electrical, mechanical, biological, acoustic, and financial domains, among many others.

► Traditionally, a signal has been defined as any quantity that exhibits a variation with either time, space, or both. Mathematically, however, variation of any quantity as a function of any independent variable would qualify as a signal as well. ◄



Acoustic pressure waveform

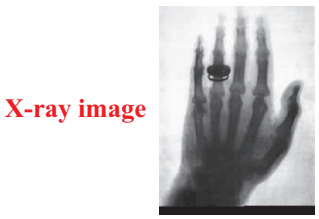
(a) Continuous-time signal




(b) Discrete-spatial signal




(c) Independent variable is age group




(d) 2-D spatial signal

**Figure 1-5:** Examples of continuous and discrete signals.

In contrast with the continuous-time signal shown in **Fig. 1-5(a)**, the brightness variation across the row of pixels on the computer display of **Fig. 1-5(b)** constitutes a ***discrete-space signal*** because the brightness is specified at only a set of discrete locations. In either case, the signal may represent a physical quantity, such as the altitude profile of atmospheric temperature, a time record of blood pressure, or fuel consumption per kilometer as a function of car speed— wherein each is plotted as a function of an independent variable—or it may represent a non-physical quantity such as a stock market index or the distribution of unemployed workers by age group (**Fig. 1-5(c)**). Moreover, in some cases, a signal may be a function of two or more variables, as illustrated by the two-dimensional (2-D) X-ray image in **Fig. 1-5(d)**.

## 1-1.2 Causal vs. Noncausal

Real systems—as opposed to purely conceptual or mathematical constructs that we may use as learning tools even though we know they cannot be realized in practice—are called ***physically realizable systems***. When such a system is excited by an input signal $x(t)$, we usually define the time dimension such that $t = 0$ coincides with when the signal is first introduced. Accordingly, $x(t) = 0$ for $t < 0$, and $x(t)$ is called a ***causal signal***. By extension, if $x(t) \neq 0$ for $t < 0$, it is called ***noncausal***, and if $x(t) = 0$ for $t > 0$, it is called ***anticausal***.

▶ A signal $x(t)$ is:

- **causal** if $x(t) = 0$ for $t < 0$ (starts at or after $t = 0$)
- **noncausal** if $x(t) \neq 0$ for any $t < 0$ (starts before $t = 0$)
- **anticausal** if $x(t) = 0$ for $t > 0$ (ends at or before $t = 0$) ◀

Even though (in practice) our ultimate goal is to evaluate the interaction of causal signals with physically realizable systems, we will occasionally use mathematical techniques that represent a causal signal in terms of artificial constructs composed of sums and differences of causal and anticausal signals.

## 1-1.3 Analog vs. Digital

Consider an electronic sensor designed such that its output voltage $v$ is linearly proportional to the air temperature $T$ surrounding its temperature-sensitive thermistor. If the sensor's output is recorded continuously as a function of time (**Fig. 1-6(b)**), the resulting voltage record $v(t)$ would be *analogous* to the pattern of the actual air temperature $T(t)$. Hence, $v(t)$ is regarded as an ***analog signal*** representing $T(t)$. The term analog (short for analogue) conveys the similarity between the measured signal and the physical quantity it represents. It also implies that because both $v$ and $t$ are continuous variables, the resolution associated with the recorded $v(t)$ is infinite along both dimensions.

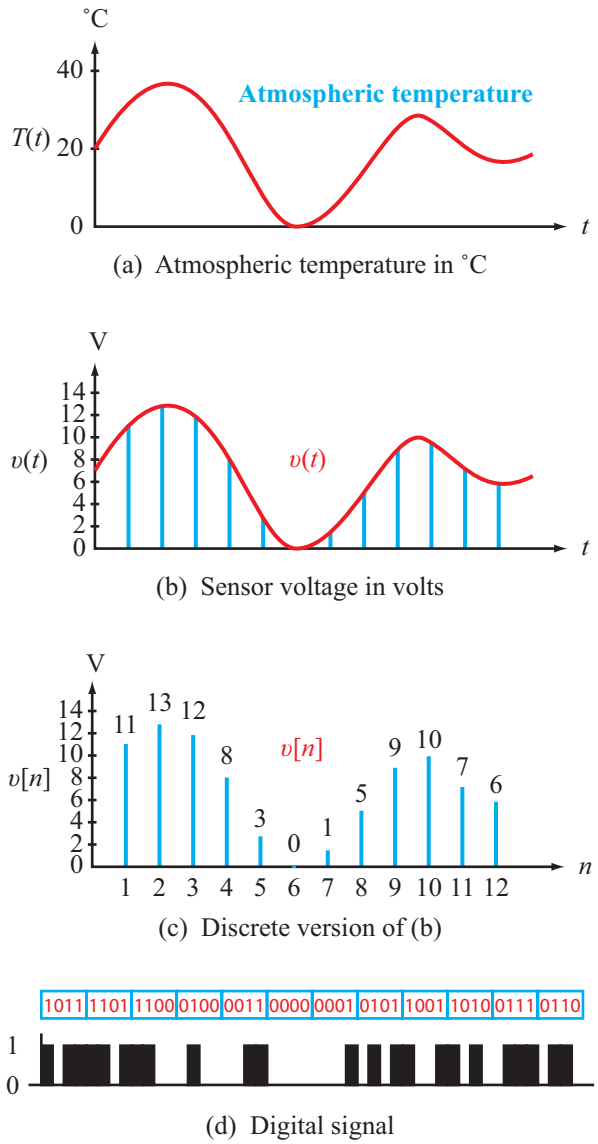



**Figure 1-6:** The atmospheric temperature temporal profile in (a) is represented in (b) by the continuous signal $\upsilon(t)$ measured by a temperature sensor. The regularly spaced sequence $\upsilon[n]$ in (c) is the discrete version of $\upsilon(t)$. The discrete signal $\upsilon[n]$ is converted into a digital sequence in (d) using a 4-bit encoder.

Had the temperature sensor recorded $\upsilon$ at only a set of equally spaced, discrete values of time, the outcome would have looked like the ***discrete-time signal*** $\upsilon[n]$ displayed in **Fig. 1-6(c)**, in which the dependent variable $\upsilon$ continues to enjoy infinite resolution in terms of its own magnitude but not along the independent variable $t$.

► To distinguish between a continuous-time signal $\upsilon(t)$ and a discrete-time signal $\upsilon[n]$, the independent variable $t$ in $\upsilon(t)$ is enclosed in curved brackets, whereas for discrete-time signal $\upsilon[n]$, the index $n$ is enclosed in square brackets. ◄

If, in addition to discretizing the signal in time, we were to quantize its amplitudes $\upsilon[n]$ using a 4-bit encoder, for example, we would end up with the ***digital discrete-time signal*** shown in **Fig. 1-6(d)**. By so doing, we have sacrificed resolution along both dimensions, raising the obvious question: *Why is it that the overwhelming majority of today's electronic and mechanical systems—including cell phones and televisions—perform their signal conditioning and display functions in the digital domain?* The most important reason is so that signal processing can be implemented on a digital computer. Computers process finite sequences of numbers, each of which is represented by a finite number of bits. Hence, to process a signal using a digital computer, it must be in discrete-time format, and its amplitude must be encoded into a binary sequence.

Another important reason for using digital signal processing has to do with ***noise***. Superimposed on a signal is (almost always) an unwanted random fluctuation (noise) contributed by electromagnetic fields associated with devices and circuits as well as by natural phenomena (such as lightning). Digital signals are more immune to noise interference than their analog counterparts.

The terms continuous-time, discrete-time, analog, and digital can be summarized as follows:

- A signal $x(t)$ is ***analog and continuous-time*** if both $x$ and $t$ are continuous variables (infinite resolution). Most real-world signals are analog and continuous-time (Chapters 1 through 6).
- A signal $x[n]$ is ***analog and discrete-time*** if the values of $x$ are continuous but time $n$ is discrete (integer-valued). Chapters 7 through 10 deal with analog discrete-time signals.
- A signal $x[n]$ is ***digital and discrete-time*** if the values of $x$ are discrete (i.e., quantized) and time $n$ also is discrete (integer-valued). Computers store and process digital discrete-time signals. This class of signals is outside the scope of this book.
- A signal $x(t)$ is ***digital and continuous-time*** if $x(t)$ can only take on a finite number of values. An example is the class of logic signals, such as the output of a flip-flop, which can only take on values 0 or 1.

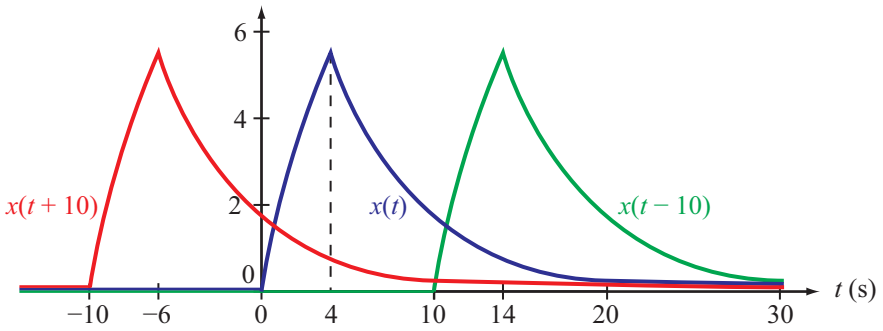



**Figure 1-7:** Waveforms of $x(t)$, $x(t-10)$, and $x(t+10)$. Note that $x(t-10)$ reaches its peak value 10 s later than $x(t)$, and $x(t+10)$ reaches its peak value 10 s sooner than $x(t)$.

As was stated earlier, a signal's independent variable may not always be time $t$, and in some cases, the signal may depend on more than one variable (as in 2-D images and 3-D X-ray tomographs). Nevertheless, in the interest of brevity when introducing mathematical techniques, we will use the symbol $t$ as our independent variable exclusively. This does not preclude using other, more appropriate, symbols when applying the techniques to specific applications, nor does it limit expanding the formulation to 2-D or 3-D when necessary.

**Concept Question 1-1:** What is the difference between a continuous-time signal and a discrete-time signal? Between a discrete-time signal and a digital signal? (See S²)

**Concept Question 1-2:** What is the definition of a *causal* signal? *Noncausal* signal? *Anticausal* signal? (See S²)

## 1-2 Signal Transformations

▶ A system transforms an input signal into an output signal. The transformation may entail the modification of some attribute of the input signal, the generation of an entirely new signal (based on the input signal), or the extraction of information from the input signal for display or to initiate an action. ◀

For example, the system may delay, compress or stretch out the input signal, or it may filter out the noise accompanying it. If the signal represents the time profile of a car's acceleration $a(t)$, as measured by a microelectromechanical sensor, the system may perform an integration to determine the car's velocity: $v(t) = \int_0^t a(\tau)\, d\tau$. In yet another example, the system may be an algorithm that generates signals to control the movements of a manufacturing robot, using the information extracted from multiple input signals.

### 1-2.1 Time-Shift Transformation

If $x(t)$ is a continuous-time signal, a ***time-shifted*** version with delay $T$ is given by

$$y(t) = x(t - T), \tag{1.1}$$

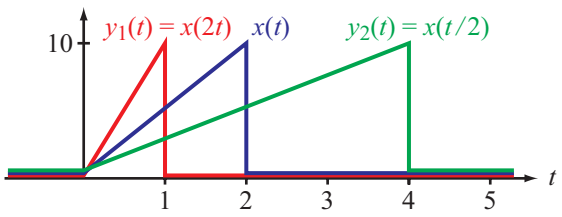
wherein $t$ is replaced with $(t - T)$ everywhere in the expression and/or plot of $x(t)$, as illustrated in **Fig. 1-7**. If $T > 0$, $y(t)$ is *delayed* by $T$ seconds relative to $x(t)$; the peak value of the waveform of $y(t)$ occurs $T$ seconds later in time than does the peak of $x(t)$. Conversely, if $T < 0$, $y(t)$ is *advanced* by $T$ seconds relative to $x(t)$, in which case the peak value of $y(t)$ occurs earlier in time.

▶ While preserving the shape of the signal $x(t)$, the ***time-shift transformation*** $x(t - T)$ is equivalent to sliding the waveform to the right along the time axis when $T$ is positive and sliding it to the left when $T$ is negative. ◀

### 1-2.2 Time-Scaling Transformation

**Figure 1-8** displays three waveforms that are all similar (but not identical) in shape. Relative to the waveform of $x(t)$, the waveform of $y_1(t)$ is compressed along the time axis, while that of $y_2(t)$ is expanded (stretched out). Waveforms of signals $y_1(t)$ and $y_2(t)$ are *time-scaled* versions of $x(t)$:

$$y_1(t) = x(2t), \tag{1.2a}$$




$$x(t) = \begin{cases} 5t & \text{for } 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$y_1(t) = \begin{cases} 5 \cdot 2t = 10t & \text{for } 0 \le 2t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$y_2(t) = \begin{cases} 5t/2 = 2.5t & \text{for } 0 \le t/2 \le 2 \\ 0 & \text{otherwise} \end{cases}$$

**Figure 1-8:** Waveforms of $x(t)$, a compressed replica given by $y_1(t) = x(2t)$, and an expanded replica given by $y_2(t) = x(t/2)$.

and

$$y_2(t) = x(t/2). \tag{1.2b}$$

Mathematically, the ***time-scaling transformation*** can be expressed as

$$y(t) = x(at), \tag{1.3}$$

where $a$ is a compression or expansion factor depending on whether its absolute value is larger or smaller than 1, respectively. For the time being, we will assume $a$ to be positive. As we will see shortly in the next subsection, a negative value of $a$ causes a time-reversal transformation in addition to the compression/expansion transformation.

► Multiplying the independent variable $t$ in $x(t)$ by a constant coefficient $a$ results in a temporally compressed replica of $x(t)$ if $|a| > 1$ and by a temporally expanded replica if $|a| < 1$. ◄

## 1-2.3 Time-Reversal Transformation

► Replacing $t$ with $-t$ in $x(t)$ generates a signal $y(t)$ whose waveform is the mirror image of that of $x(t)$ with respect to the vertical axis. ◄

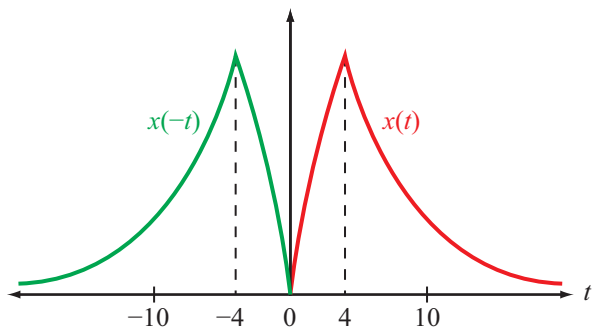



**Figure 1-9:** Waveforms of $x(t)$ and its time reversal $x(-t)$.

The ***time-reversal transformation*** is expressed as

$$y(t) = x(-t), \tag{1.4}$$

and is illustrated by the waveforms in **Fig. 1-9**.

## 1-2.4 Combined Transformation

The three aforementioned transformations can be combined into a generalized transformation:

$$y(t) = x(at - b) = x\left(a\left(t - \frac{b}{a}\right)\right) = x(a(t - T)), \tag{1.5}$$

where $T = b/a$. We recognize $T$ as the time shift and $a$ as the compression/expansion factor. Additionally, the sign of $a$ ($-$ or $+$) denotes whether or not the transformation includes a time-reversal transformation.

The procedure for obtaining $y(t) = x(a(t - T))$ from $x(t)$ is as follows:

**(1) Scale time by $a$:**

- If $|a| < 1$, then $x(t)$ is expanded.
- If $|a| > 1$, then $x(t)$ is compressed.
- If $a < 0$, then $x(t)$ is also reflected.

This results in $z(t) = x(at)$.

**(2) Time shift by $T$:**

- If $T > 0$, then $z(t)$ shifts to the right.
- If $T < 0$, then $z(t)$ shifts to the left.

This results in $z(t - T) = x(a(t - T)) = y(t)$.

The procedure for obtaining $y(t) = x(at - b)$ from $x(t)$ reverses the order of time scaling and time shifting:

(1) Time shift by $b$.

(2) Time scale by $a$.

**Example 1-1: Multiple Transformations**

For signal $x(t)$ profiled in **Fig. 1-10(a)**, generate the corresponding profile of $y(t) = x(-2t + 6)$.

**Solution:** We start by recasting the expression for the dependent variable into the standard form given by Eq. (1.5),

$$y(t) = x\left(-2\left(t - \frac{6}{2}\right)\right)$$
$$= x(-2(t - 3)).$$

Reversal Compression factor Time-shift

We need to apply the following transformations:

**(1) Scale time by $-2t$:** This causes the waveform to reflect around the vertical axis and then compresses time by a factor of 2. These steps can be performed in either order. The result, $z(t) = x(-2t)$, is shown in **Fig. 1-10(b)**.

**(2) Delay waveform $z(t)$ by 3 s:** This shifts the waveform to the right by 3 s (because the sign of the time shift is negative). The result, $y(t) = z(t - 3) = x(-2(t - 3))$, is displayed in **Fig. 1-10(c)**.

**Concept Question 1-3:** Is the shape of a waveform altered or preserved upon applying a time-shift transformation? Time-scaling transformation? Time-reversal transformation? (See S²)

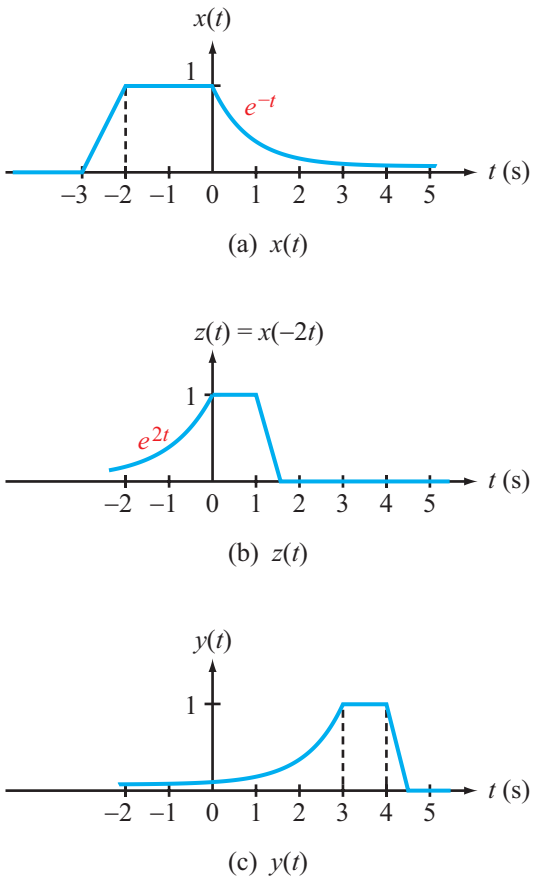



**Figure 1-10:** Waveforms of Example 1-1.

**Exercise 1-1:** If signal $y(t)$ is obtained from $x(t)$ by applying the transformation $y(t) = x(-4t - 8)$, determine the values of the transformation parameters $a$ and $T$.

**Answer:** $a = -4$ and $T = -2$. (See S²)

**Exercise 1-2:** If $x(t) = t^3$ and $y(t) = 8t^3$, are $x(t)$ and $y(t)$ related by a transformation?

**Answer:** Yes, because $y(t) = 8t^3 = (2t)^3 = x(2t)$. (See S²)

**Exercise 1-3:** What type of transformations connect $x(t) = 4t$ to $y(t) = 2(t + 4)$?

**Answer:** $y(t) = x(\frac{1}{2}(t + 4))$, which includes a time-scaling transformation with a factor of 1/2 and a time-shift transformation with a time advance of 4 s. (See S²)

## 1-3 Waveform Properties

### 1-3.1 Even Symmetry

▶ A signal $x(t)$ exhibits ***even symmetry*** if its waveform is symmetrical with respect to the vertical axis. ◀

The shape of the waveform on the left-hand side of the vertical axis is the mirror image of the waveform on the right-hand side. Mathematically, a signal $x(t)$ has *even symmetry* if

$$x(t) = x(-t) \qquad \textbf{(even symmetry)}. \tag{1.6}$$

A signal has even symmetry if reflection about the vertical axis leaves its waveform unaltered.

The signal displayed in **Fig. 1-11(b)** is an example of a waveform that exhibits even symmetry. Other examples include $\cos(\omega t)$ and $t^n$ for even integers $n$.

### 1-3.2 Odd Symmetry

In contrast, the waveform in **Fig. 1-11(c)** has ***odd symmetry***.

▶ A signal exhibits odd symmetry if the shape of its waveform on the left-hand side of the vertical axis is the *inverted* mirror image of the waveform on the right-hand side. ◀

Equivalently,

$$x(t) = -x(-t) \qquad \textbf{(odd symmetry)}. \tag{1.7}$$

A signal has odd symmetry if reflection about the vertical axis, followed by reflection about the horizontal axis, leaves its waveform unaltered. Examples of odd signals include $\sin(\omega t)$ and $t^n$ for odd integers $n$.

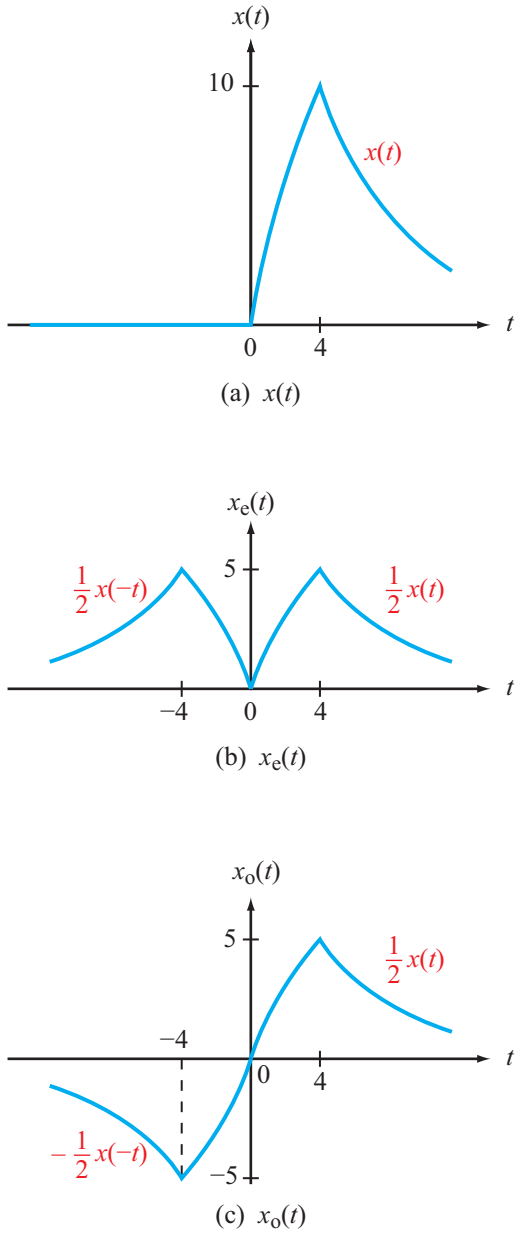



**Figure 1-11:** Signal $x(t)$ and its even and odd components.

We should note that if signal $y(t)$ is equal to the product of two signals, namely

$$y(t) = x_1(t)\, x_2(t), \tag{1.8}$$

then $y(t)$ will exhibit even symmetry if $x_1(t)$ and $x_2(t)$ both exhibit the same type of symmetry (both even or both odd)

**Table 1-1: Signal transformations.**

| Transformation | Expression | Consequence |
|---|---|---|
| **Time shift** | $y(t) = x(t - T)$ | Waveform is shifted along $+t$ direction if $T > 0$ and along $-t$ direction if $T < 0$. |
| **Time scaling** | $y(t) = x(at)$ | Waveform is compressed if $\|a\| > 1$ and expanded if $\|a\| < 1$. |
| **Time reversal** | $y(t) = x(-t)$ | Waveform is mirror-imaged relative to vertical axis. |
| **Generalized** | $y(t) = x(at - b)$ | Shift by $b$, then scale by $a$, time reversal if $a < 0$. |
| **Even / odd synthesis** | $x(t) = x_e(t) + x_o(t)$ | $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$, $x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$. |

and $y(t)$ will exhibit odd symmetry if $x_1(t)$ and $x_2(t)$ exhibit different forms of symmetry. That is,

$$\text{(even)} \times \text{(even)} = \text{even},$$
$$\text{(even)} \times \text{(odd)} = \text{odd},$$

and

$$\text{(odd)} \times \text{(odd)} = \text{even}.$$

### 1-3.3 Even / Odd Synthesis

In Chapter 5, we will find it easier to analyze a signal if it possesses even or odd symmetry than if it possesses neither. In that case, it may prove advantageous to synthesize a signal $x(t)$ as the sum of two component signals, one with even symmetry and another with odd symmetry:

$$x(t) = x_e(t) + x_o(t), \tag{1.9}$$

with

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)], \tag{1.10a}$$
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]. \tag{1.10b}$$

As the graphical example shown in **Fig. 1-11** demonstrates, adding a copy of its time reversal, $x(-t)$, to any signal $x(t)$ generates a signal with even symmetry. Conversely, subtracting a copy of its time reversal from $x(t)$ generates a signal with odd symmetry.

**Table 1-1** provides a summary of the linear transformations examined thus far.

### 1-3.4 Periodic vs. Nonperiodic

A signal's waveform may be ***periodic*** or ***nonperiodic*** (also called ***aperiodic***).

► A periodic signal $x(t)$ of period $T_0$ satisfies the ***periodicity*** property:

$$x(t) = x(t + nT_0) \tag{1.11}$$

for all integer values of $n$ and all times $t$. ◄

The periodicity property states that the waveform of $x(t)$ repeats itself every $T_0$ seconds. Examples of periodic signals are displayed in **Fig. 1-12**.

Note that if a signal is periodic with period $T_0$, it is also periodic with period $2T_0$, $3T_0$, etc. The ***fundamental period*** of a periodic signal is the smallest value of $T_0$ such that Eq. (1.11) is satisfied for all integer values of $n$. In future references, the term "period" shall refer to the fundamental period $T_0$.

The most important family of periodic signals are sinusoids. A sinusoidal signal $x(t)$ has the form

$$x(t) = A\cos(\omega_0 t + \theta), \qquad -\infty < t < \infty,$$

where

$$A = \text{amplitude}; \quad x_{\max} = A \quad \text{and} \quad x_{\min} = -A,$$
$$\omega_0 = \text{angular frequency in rad/s},$$
$$\theta = \text{phase-angle shift in radians or degrees}.$$

Related quantities include

$$f_0 = \omega_0/2\pi = \text{circular frequency in Hertz},$$
$$T_0 = 1/f_0 = \text{period of } x(t) \text{ in seconds}.$$

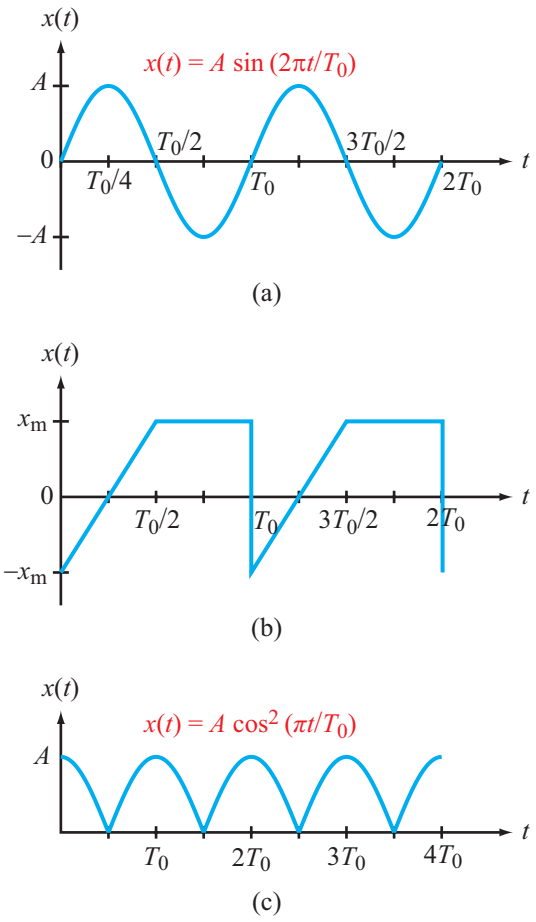



**Figure 1-12:** Examples of periodic waveforms.

Another important periodic signal is the complex exponential given by

$$x(t) = Ae^{j\omega_0 t} = |A|e^{j(\omega_0 t+\theta)},$$

where, in general, $A$ is a complex amplitude given by

$$A = |A|e^{j\theta}.$$

By Euler's formula,

$$x(t) = |A|e^{j(\omega_0 t+\theta)} = |A|\cos(\omega_0 t + \theta) + j|A|\sin(\omega_0 t + \theta).$$

Hence, the complex exponential is periodic with period $T_0 = 2\pi/\omega_0$.

**Concept Question 1-4:** Define *even-symmetrical* and *odd-symmetrical* waveforms. (See S²)

**Concept Question 1-5:** State the periodicity property. (See S²)

**Exercise 1-4:** Which of the following functions have even-symmetrical waveforms, odd-symmetrical waveforms, or neither? (a) $x_1(t) = 3t^2$, (b) $x_2(t) = \sin(2t)$, (c) $x_3(t) = \sin^2(2t)$, (d) $x_4(t) = 4e^{-t}$, (e) $x_5(t) = |\cos 2t|$.

**Answer:** (a), (c), and (e) have even symmetry; (b) has odd symmetry; (d) has no symmetry. (See S²)

## 1-4 Nonperiodic Waveforms

▶ A nonperiodic signal is any signal that does not satisfy the periodicity property. ◀

Many real-world signals are often modeled in terms of a core set of elementary waveforms which includes the step, ramp, pulse, impulse, and exponential waveforms, and combinations thereof. Accordingly, we will use this section to review their properties and mathematical expressions and to point out the connections between them.

### 1-4.1 Step-Function Waveform

The waveform of signal $u(t)$ shown in **Fig. 1-13(a)** is an (ideal) ***unit step function***: It is equal to zero for $t < 0$, at $t = 0$ it makes a *discontinuous* jump to 1, and from there on forward it remains at 1. Mathematically, $u(t)$ is defined as

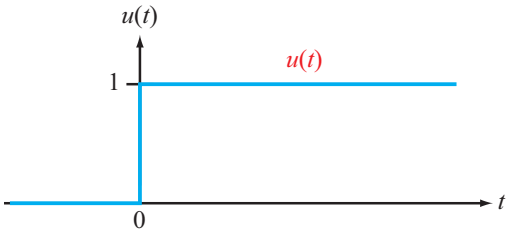
$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t > 0. \end{cases} \tag{1.12}$$

Because $u(t)$ does not have a unique value at $t = 0$, its derivative is infinite at $t = 0$, qualifying $u(t)$ as a singularity function.

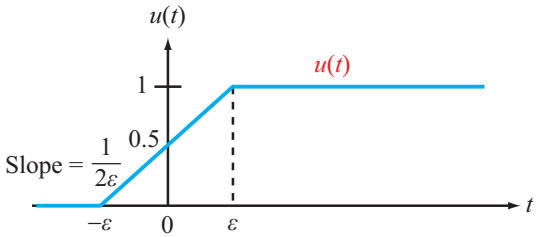
▶ A ***singularity function*** is a function such that either itself or one (or more) of its derivatives is (are) not finite everywhere. ◀

Occasionally, it may prove more convenient to model the unit step function as a ramp over an infinitesimal interval extending

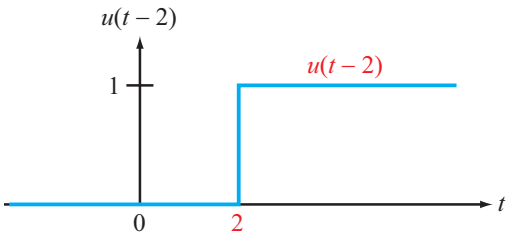
**Step Functions**



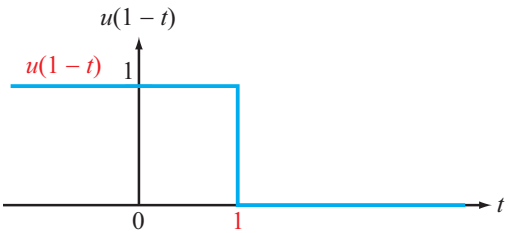

(a) $u(t)$




(b) Gradual step model




(c) Time-shifted step function with $T = 2$.




(d) Time-reversed step function with $T = 1$.

**Figure 1-13:** Unit step function.

between $-\epsilon$ and $+\epsilon$, as shown in **Fig. 1-13(b)**. Accordingly, $u(t)$ can be defined as

$$u(t) = \lim_{\epsilon \to 0} \begin{cases} 0 & \text{for } t \le -\epsilon, \\ \left[\frac{1}{2}\left(\frac{t}{\epsilon} + 1\right)\right] & \text{for } -\epsilon \le t \le \epsilon, \\ 1 & \text{for } t \ge \epsilon. \end{cases} \tag{1.13}$$

With this alternative definition, $u(t)$ is a continuous function everywhere, but in the limit as $\epsilon \to 0$ its slope in the interval $(-\epsilon, \epsilon)$ is

$$u'(t) = \lim_{\epsilon \to 0} \frac{d}{dt}\left[\frac{1}{2}\left(\frac{t}{\epsilon} + 1\right)\right] = \lim_{\epsilon \to 0}\left(\frac{1}{2\epsilon}\right) \to \infty. \tag{1.14}$$

The slope of $u(t)$ still is not finite at $t = 0$, consistent with the formal definition given by Eq. (1.12), which describes the unit step function as making an *instantaneous* jump at $t = 0$. As will be demonstrated later in Section 1-4.4, the alternative definition for $u(t)$ given by Eq. (1.13) will prove useful in establishing the connection between $u(t)$ and the impulse function $\delta(t)$.

The ***unit time-shifted step function*** $u(t-T)$ is a step function that transitions between its two levels when its argument $(t - T) = 0$:

$$u(t - T) = \begin{cases} 0 & \text{for } t < T, \\ 1 & \text{for } t > T. \end{cases} \tag{1.15}$$

▶ For any unit step function, its value is zero when its argument is less than zero and one when its argument is greater than zero. ◀

Extending this definition to the ***time-reversed step function***, we have

$$u(T - t) = \begin{cases} 1 & \text{for } t < T, \\ 0 & \text{for } t > T. \end{cases} \tag{1.16}$$

By way of examples, **Figs. 1-13(c)** and **(d)** display plots of $u(t - 2)$ and $u(1 - t)$, respectively.

## 1-4.2 Ramp-Function Waveform

The ***unit ramp function*** $r(t)$ and the ***unit time-shifted ramp function*** $r(t - T)$ are defined as

$$r(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t & \text{for } t \ge 0, \end{cases} \tag{1.17a}$$

and

$$r(t - T) = \begin{cases} 0 & \text{for } t \le T, \\ (t - T) & \text{for } t \ge T. \end{cases} \tag{1.17b}$$

Two ramp-function examples are displayed in **Fig. 1-14**. In each case, the ramp function is zero when its argument $(t - T)$ is

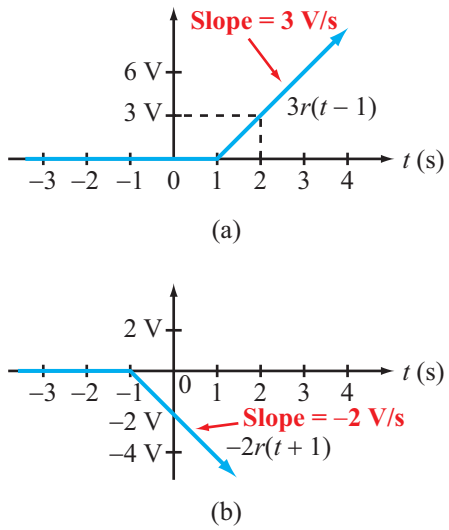



**Figure 1-14:** Examples of ramp functions.

smaller than zero, and equal to its own argument when the value of $t$ is such that the argument is greater than zero. The slope of a ramp function $x(t) = ar(t - T)$ is specified by the constant coefficient $a$.

Because the time-derivative of $r(t - T)$—i.e., its slope—is discontinuous at $t = T$, the ramp function qualifies as a ***singularity function***.

The unit ramp function is related to the unit step function by

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = t\, u(t), \tag{1.18}$$

and for the time-shifted case,

$$r(t - T) = \int_{-\infty}^{t} u(\tau - T)\, d\tau = (t - T)\, u(t - T). \tag{1.19}$$

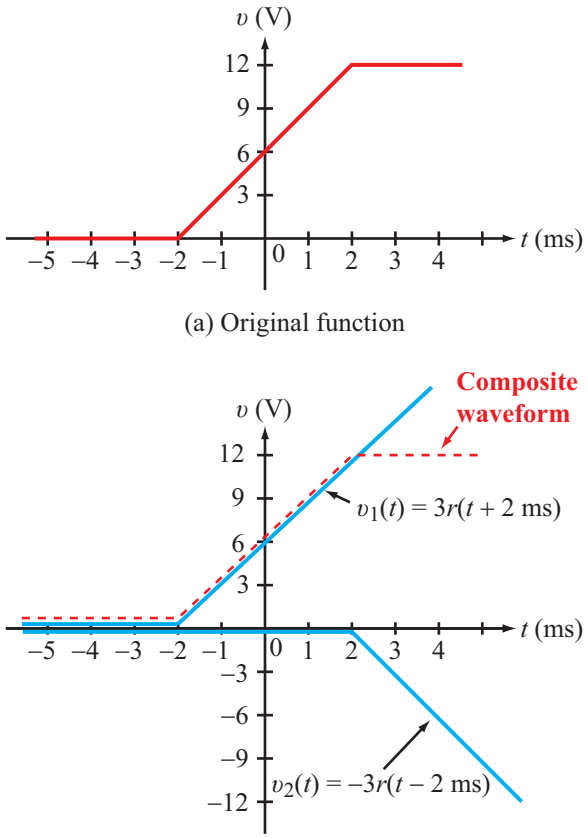



**Figure 1-15:** Step waveform of Example 1-2.

**Example 1-2: Synthesizing a Step-Waveform**

For the (realistic) step waveform $\upsilon(t)$ displayed in **Fig. 1-15**, develop expressions in terms of ramp and ideal step functions. Note that $\upsilon(t)$ is in volts (V) and the time scale is in milliseconds.

**Solution:** The voltage $\upsilon(t)$ can be synthesized as the sum of two time-shifted ramp functions (**Fig. 1-15(b)**): One starts at $t = -2$ ms and has a positive slope of 3 V/ms and a second starts at $t = 2$ ms but with a slope of $-3$ V/ms. Thus,

$$\upsilon(t) = \upsilon_1(t) + \upsilon_2(t) = 3r(t + 2\text{ ms}) - 3r(t - 2\text{ ms}) \quad \text{V}.$$

In view of Eq. (1.19), $\upsilon(t)$ also can be expressed as

$$\upsilon(t) = 3(t + 2\text{ ms})\, u(t + 2\text{ ms}) - 3(t - 2\text{ ms})\, u(t - 2\text{ ms}) \quad \text{V}.$$

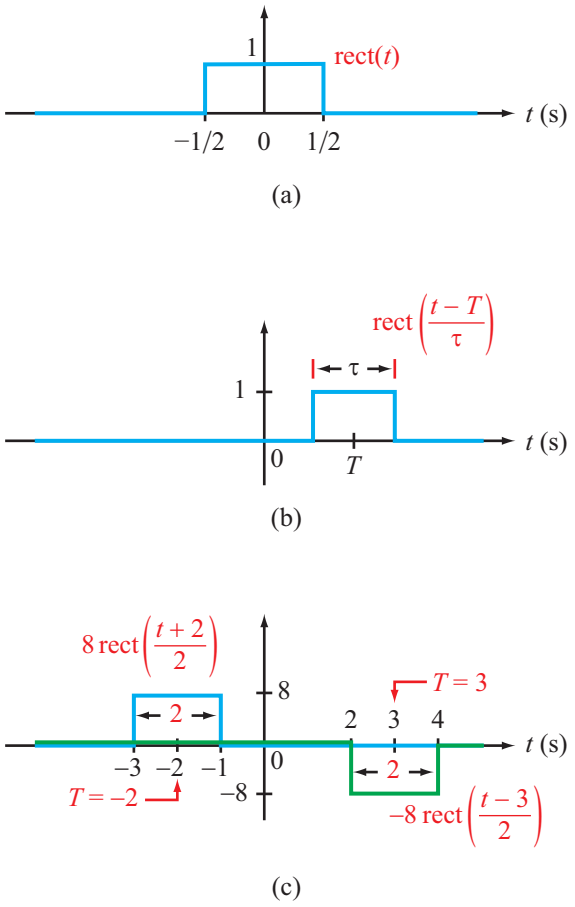



**Figure 1-16:** Rectangular pulses.

### 1-4.3 Pulse Waveform

The ***rectangular function*** rect$(t)$ is defined as

$$\text{rect}(t) = \begin{cases} 1 & \text{for } |t| < \frac{1}{2} \\ 0 & \text{for } |t| > \frac{1}{2} \end{cases} = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right),$$

and its waveform is displayed in **Fig. 1-16(a)**. Note that the rectangle is of width 1 s, height 1 unit, and is centered at $t = 0$.

In general, a rectangular pulse can be described mathematically by the rectangular function rect$[(t - T)/\tau]$. Its two parameters are $T$, which defines the location of the center of the pulse along the $t$-axis, and $\tau$, which is the duration of the pulse (**Fig. 1-16(b)**). Examples are shown in **Fig. 1-16(c)**. The general rectangular function is defined as

$$\text{rect}\left(\frac{t-T}{\tau}\right) = \begin{cases} 0 & \text{for } t < (T - \tau/2), \\ 1 & \text{for } (T - \tau/2) < t < (T + \tau/2), \\ 0 & \text{for } t > (T + \tau/2). \end{cases} \tag{1.20}$$

We note that because the rectangular function is discontinuous at its two edges (namely at $t = T - \tau/2$ and $t = T + \tau/2$), it is a bona fide member of the family of singularity functions.

**Example 1-3: Rectangular and Trapezoidal Pulses**

Develop expressions in terms of ideal step functions for (a) the rectangular pulse $v_a(t)$ in **Fig. 1-17(a)** and (b) the more-realistic trapezoidal pulse $v_b(t)$ in **Fig. 1-17(b)**.

**Solution: (a)** The amplitude of the rectangular pulse is 4 V, its duration is 2 s, and its center is at $t = 3$ s. Hence,

$$v_a(t) = 4\,\text{rect}\left(\frac{t-3}{2}\right) \quad \text{V}.$$

The sequential addition of two time-shifted step functions, $v_1(t)$ at $t = 2$ s and $v_2(t)$ at $t = 4$ s, as demonstrated graphically in **Fig. 1-17(c)**, accomplishes the task of synthesizing the rectangular pulse in terms of two step functions:

$$v_a(t) = v_1(t) + v_2(t) = 4[u(t-2) - u(t-4)] \quad \text{V}.$$

Generalizing, a unit rectangular function rect$[(t-T)/\tau]$ always can be expressed as

$$\text{rect}\left(\frac{t-T}{\tau}\right) = u\left[t - \left(T - \frac{\tau}{2}\right)\right] - u\left[t - \left(T + \frac{\tau}{2}\right)\right]. \tag{1.21}$$

**(b)** The trapezoidal pulse exhibits a change in slope at $t = 0$, $t = 1$ s, $t = 3$ s, and $t = 4$ s, each of which can be accommodated by the introduction of a time-shifted ramp function with the appropriate slope. Building on the procedure used in Example 1-2, $v_b(t)$ can be synthesized as the sum of

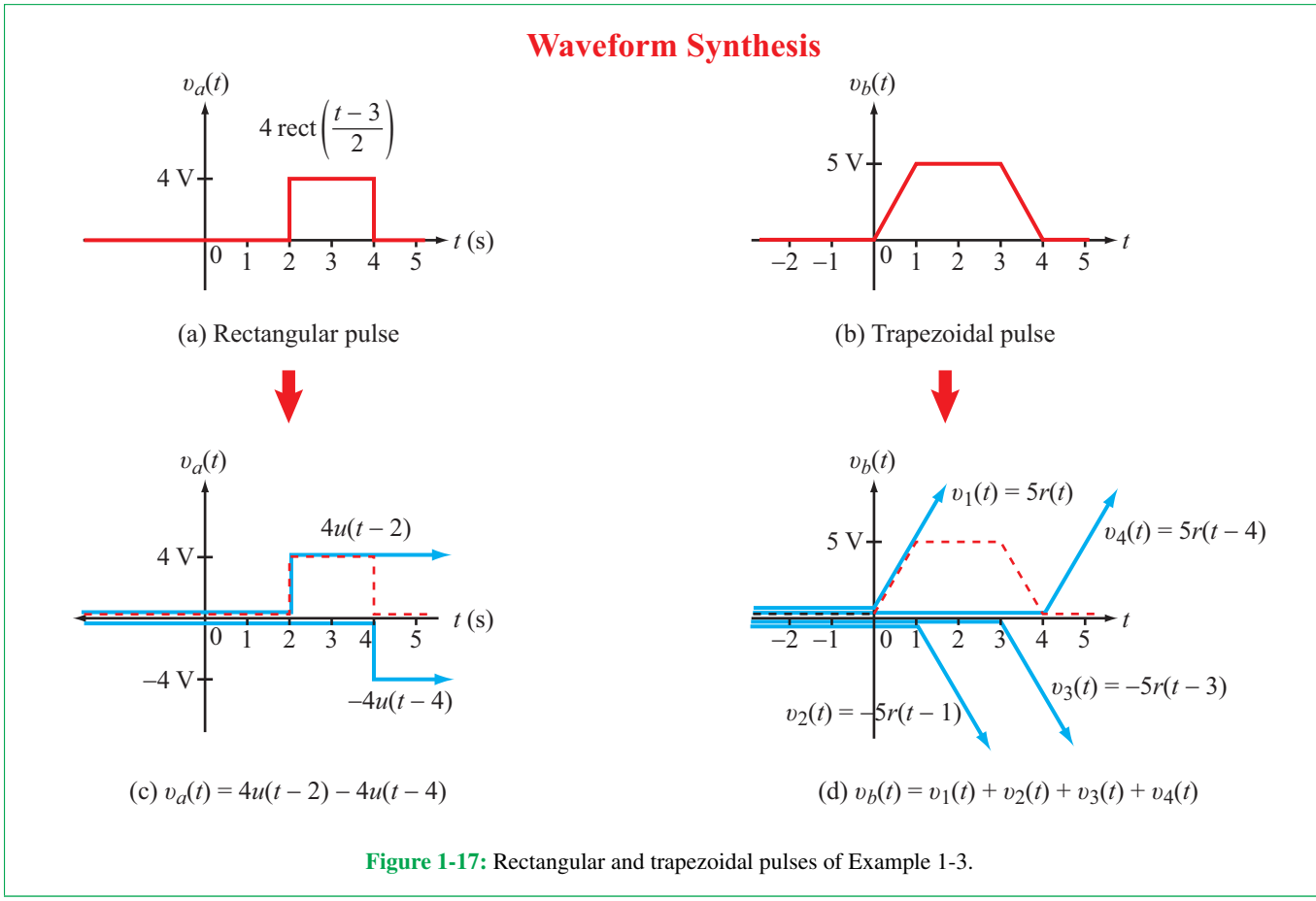



**Figure 1-17:** Rectangular and trapezoidal pulses of Example 1-3.

the four ramp functions shown in **Fig. 1-17(d)**:

$$\begin{aligned} v_b(t) &= v_1(t) + v_2(t) + v_3(t) + v_4(t) \\ &= 5[r(t) - r(t-1) - r(t-3) + r(t-4)] \\ &= 5[t\,u(t) - (t-1)\,u(t-1) \\ &\quad - (t-3)\,u(t-3) + (t-4)\,u(t-4)] \text{ V}, \end{aligned}$$

where in the last step, we used the relation given by Eq. (1.19).

**Example 1-4: Periodic Sawtooth Waveform**

Express the periodic sawtooth waveform shown in **Fig. 1-18** in terms of step and ramp functions.

**Solution:** The segment between $t = 0$ and $t = 2$ s is a ramp with a slope of 5 V/s. To effect a sudden drop from 10 V down to zero at $t = 2$ s, we need to (a) add a negative ramp function at $t = 2$ s and (b) add a negative offset of 10 V in the form of a delayed step function. Hence, for this time segment,

$$x_1(t) = [5r(t) - 5r(t-2) - 10u(t-2)] \text{ V}, \quad 0 \le t < 2 \text{ s}.$$

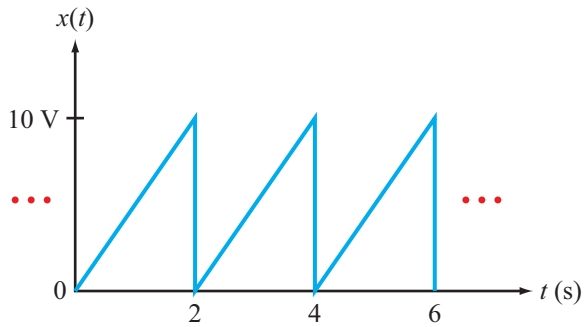



**Figure 1-18:** Periodic sawtooth waveform of Example 1-4.

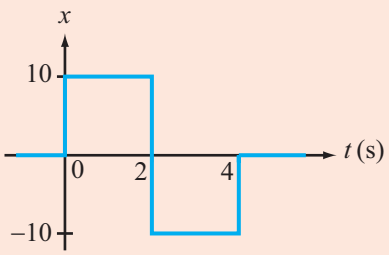
By extension, for the entire periodic sawtooth waveform with period $T_0 = 2$ s, we have

$$x(t) = \sum_{n=-\infty}^{\infty} x_1(t - nT_0) = \sum_{n=-\infty}^{\infty} [5r(t-2n) - 5r(t-2-2n) - 10u(t-2-2n)] \text{ V}.$$

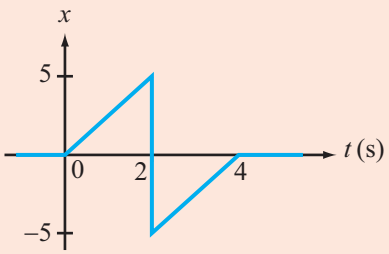
**Concept Question 1-6:** How are the ramp and rectangle functions related to the step function? (See S²)

**Concept Question 1-7:** The step function $u(t)$ is considered a singularity function because it makes a discontinuous jump at $t = 0$. The ramp function $r(t)$ is continuous at $t = 0$. Yet it also is a singularity function. Why? (See S²)

**Exercise 1-5:** Express the waveforms shown in **Fig. E1-5** in terms of unit step or ramp functions.



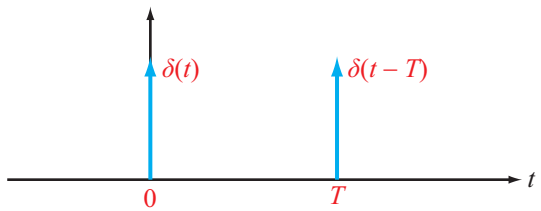

(a)



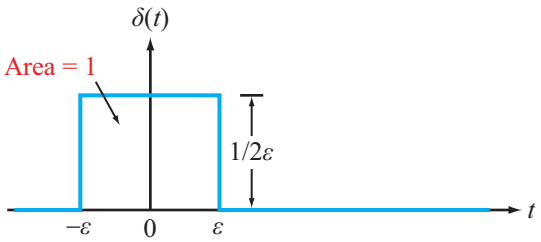

(b)

**Figure E1-5**

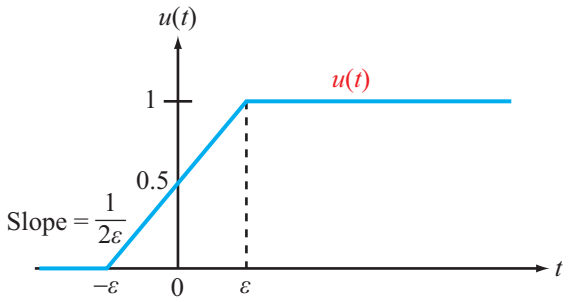
**Answer:** (a) $x(t) = 10\,u(t) - 20\,u(t-2) + 10\,u(t-4)$, (b) $x(t) = 2.5\,r(t) - 10\,u(t-2) - 2.5\,r(t-4)$. (See S²)




(a) $\delta(t)$ and $\delta(t - T)$




(b) Rectangle model for $\delta(t)$




(c) Gradual step model for $u(t)$

**Figure 1-19:** Unit impulse function.

**Exercise 1-6:** How is $u(t)$ related to $u(-t)$?

**Answer:** They are mirror images of one another (with respect to the $y$-axis). (See S²)

## 1-4.4 Impulse Function

Another member of the family of singularity functions is the ***unit impulse function***, which is also known as the ***Dirac delta function*** $\delta(t)$. Graphically, it is represented by a vertical arrow, as shown in **Fig. 1-19(a)**. If its location is time-shifted to $t = T$, it is designated $\delta(t - T)$. For any specific location $T$, the

impulse function is defined through the combination of two properties:

$$\delta(t - T) = 0 \qquad \text{for } t \neq T \tag{1.22a}$$

and

$$\int_{-\infty}^{\infty} \delta(t - T)\, dt = 1. \tag{1.22b}$$

▶ The first property states that the impulse function $\delta(t - T)$ is zero everywhere, except at its own location ($t = T$), but its value is infinite at that location. The second property states that the total area under the unit impulse is equal to 1, regardless of its location. ◀

To appreciate the meaning of the second property, we can represent the impulse function by the rectangle shown in **Fig. 1-19(b)** with the understanding that $\delta(t)$ is defined in the limit as $\epsilon \to 0$. The rectangle's dimensions are such that its width, $w = 2\epsilon$, and height, $h = 1/(2\epsilon)$, are reciprocals of one another. Consequently, the area of the rectangle is always unity, even as $\epsilon \to 0$.

According to the rectangle model displayed in **Fig. 1-19(b)**, $\delta(t) = 1/(2\epsilon)$ over the narrow range $-\epsilon < t < \epsilon$. For the gradual step model of $u(t)$ shown in **Fig. 1-19(c)**, its slope also is $1/(2\epsilon)$. Hence,

$$\frac{du(t)}{dt} = \delta(t). \tag{1.23}$$

Even though this relationship between the unit impulse and unit step functions was obtained on the basis of specific geometrical models for $\delta(t)$ and $u(t)$, its validity can be demonstrated to be true always. The corresponding expression for $u(t)$ is

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau, \tag{1.24}$$

and for the time-shifted case,

$$\frac{d}{dt}\,[u(t - T)] = \delta(t - T), \tag{1.25a}$$

$$u(t - T) = \int_{-\infty}^{t} \delta(\tau - T)\, d\tau. \tag{1.25b}$$

By extension, a scaled impulse $k\,\delta(t)$ has an area $k$ and

$$\int_{-\infty}^{t} k\,\delta(\tau)\, d\tau = k\,u(t). \tag{1.26}$$

**Example 1-5: Alternative Models for Impulse Function**

Show that models $x_1(t)$ and $x_2(t)$ in **Fig. 1-20** qualify as unit impulse functions in the limit as $\epsilon \to 0$.

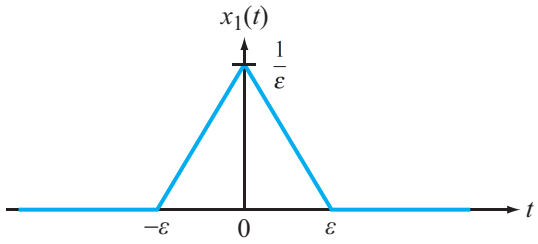
**Solution:** To qualify as a unit impulse function, a function must: (1) be zero everywhere except at $t = 0$, (2) be infinite at $t = 0$, (3) be even, and (4) have a unit area.

(a) Triangle Model $x_1(t)$

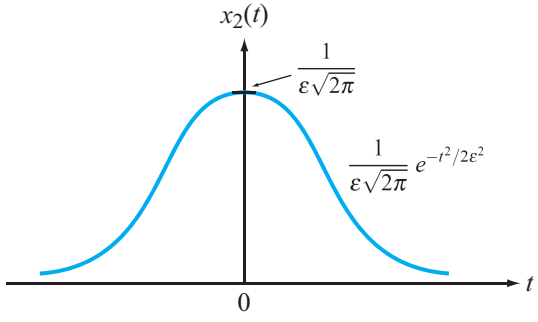
**(1)** As $\epsilon \to 0$, $x_1(t)$ is indeed zero everywhere except at $t = 0$.

**(2)** $\lim_{\epsilon \to 0} x_1(0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} = \infty$; hence infinite at $t = 0$.

**(3)** $x_1(t)$ is clearly an even function (**Fig. 1-20(a)**).




(a) Triangle model $x_1(t)$




(b) Gaussian model $x_2(t)$

**Figure 1-20:** Alternative models for $\delta(t)$.

**(4)** Area of triangle $= \frac{1}{2}\left(2\epsilon \times \frac{1}{\epsilon}\right) = 1$, regardless of the value of $\epsilon$.

Hence, $x_1(t)$ does qualify as a unit impulse function.

(b) Gaussian Model $x_2(t)$

**(1)** Except at $t = 0$, as $\epsilon \to 0$, the magnitude of the exponential $e^{-t^2/2\epsilon^2}$ always will be smaller than $\epsilon\sqrt{2\pi}$. Hence, $x_2(t) \to 0$ as $\epsilon \to 0$, except at $t = 0$.

**(2)** At $t = 0$,

$$\lim_{\epsilon\to 0}\left[\frac{1}{\epsilon\sqrt{2\pi}}\, e^{-t^2/2\epsilon^2}\right]_{t=0} = \lim_{\epsilon\to 0}\left[\frac{1}{\epsilon\sqrt{2\pi}}\right] = \infty.$$

**(3)** $x_2(t)$ is clearly an even function (**Fig. 1-20(b)**).

**(4)** The area of the Gaussian model is

$$A = \int_{-\infty}^{\infty} \frac{1}{\epsilon\sqrt{2\pi}}\, e^{-t^2/2\epsilon^2}\, dt.$$

Applying the integral formula

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{a}$$

leads to $A = 1$. Hence, $x_2(t)$ qualifies as a unit impulse function.

## 1-4.5 Sampling Property of $\delta(t)$

As was noted earlier, multiplying an impulse function by a constant $k$ gives a scaled impulse of area $k$. Now we consider what happens when a continuous-time function $x(t)$ is multiplied by $\delta(t)$. Since $\delta(t)$ is zero everywhere except at $t = 0$, it follows that

$$x(t)\,\delta(t) = x(0)\,\delta(t), \tag{1.27}$$

provided that $x(t)$ is continuous at $t = 0$. By extension, multiplication of $x(t)$ by the time-shifted impulse function $\delta(t - T)$ gives

$$x(t)\,\delta(t - T) = x(T)\,\delta(t - T). \tag{1.28}$$

► Multiplication of a time-continuous function $x(t)$ by an impulse located at $t = T$ generates a scaled impulse of magnitude $x(T)$ at $t = T$, provided $x(t)$ is continuous at $t = T$. ◄

One of the most useful features of the impulse function is its ***sampling (or sifting) property***. For any function $x(t)$ known to be continuous at $t = T$:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t - T)\, dt = x(T). \tag{1.29}$$

**(sampling property)**

Derivation of the sampling property relies on Eqs. (1.22b) and (1.28):

$$\int_{-\infty}^{\infty} x(t)\,\delta(t - T)\, dt = \int_{-\infty}^{\infty} x(T)\,\delta(t - T)\, dt$$

$$= x(T)\int_{-\infty}^{\infty} \delta(t - T)\, dt = x(T).$$

## 1-4.6 Time-Scaling Transformation of $\delta(t)$

To determine how time scaling affects impulses, let us evaluate the area of $\delta(at)$:

$$\int_{-\infty}^{\infty} \delta(at)\, dt = \int_{-\infty}^{\infty} \delta(\tau)\, \frac{d\tau}{|a|} = \frac{1}{|a|}.$$

Hence, $\delta(at)$ is an impulse of area $1/|a|$. It then follows that

$$\delta(at) = \frac{1}{|a|}\,\delta(t) \qquad \text{for } a \neq 0. \tag{1.30}$$

**(time-scaling property)**

This result can be visualized for $|a| > 1$ by recalling that scaling time by $a$ compresses the time axis by $|a|$. The area of the uncompressed rectangle in **Fig. 1-19(b)** is

$$\text{Area of } \delta(t): \qquad \frac{1}{2\epsilon}[\epsilon - (-\epsilon)] = 1.$$

Repeating the calculation for a compressed rectangle gives

$$\text{Area of } \delta(at): \qquad \frac{1}{2\epsilon}\left[\frac{\epsilon}{|a|} - \left(-\frac{\epsilon}{|a|}\right)\right] = \frac{1}{|a|}.$$

Also note that the impulse is an even function because

$$\delta(-t) = \frac{1}{|-1|}\,\delta(t) = \delta(t),$$

$$\delta(at - b) = \frac{1}{|a|}\,\delta\left(t - \frac{b}{a}\right).$$

**Example 1-6: Impulse Integral**

Evaluate $\int_1^2 t^2\,\delta(2t-3)\,dt$.

**Solution:** Using the time-scaling property, the impulse function can be expressed as

$$\delta(2t-3) = \delta\left(2\left(t - \frac{3}{2}\right)\right) = \frac{1}{2}\,\delta\left(t - \frac{3}{2}\right).$$

Hence,

$$\int_1^2 t^2\,\delta(2t-3)\,dt = \frac{1}{2}\int_1^2 t^2\,\delta\left(t - \frac{3}{2}\right)dt = \frac{1}{2}\left(\frac{3}{2}\right)^2 = \frac{9}{8}.$$

We note that $\delta(t-(3/2)) \neq 0$ only at $t = 3/2$, which is included in the interval of integration, $1 \leq t \leq 2$.

**Concept Question 1-8:** How is $u(t)$ related to $\delta(t)$? (See S²)

**Concept Question 1-9:** Why is Eq. (1.29) called the *sampling property* of the impulse function? (See S²)

**Exercise 1-7:** If $x(t)$ is the rectangular pulse shown in **Fig. E1-7(a)**, determine its time derivative $x'(t)$ and plot it.

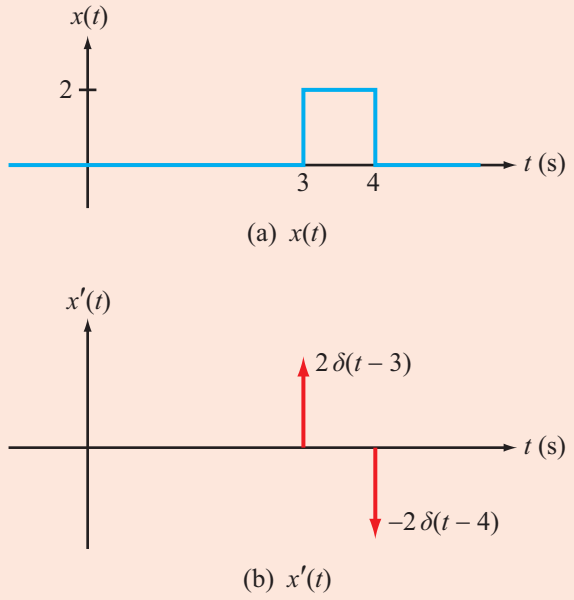



**Figure E1-7**

**Answer:** $x'(t) = 2\delta(t-3) - 2\delta(t-4)$. (See S²)

## 1-4.7 Exponential Waveform

The ***exponential function*** is a particularly useful tool for characterizing fast-rising and fast-decaying waveforms. **Figure 1-21** displays plots for

$$x_1(t) = e^{t/\tau}$$

and

$$x_2(t) = e^{-t/\tau}$$

for $\tau > 0$. The rates of increase of $x_1(t)$ and decrease of $x_2(t)$ are governed by the magnitude of the ***time constant*** $\tau$.

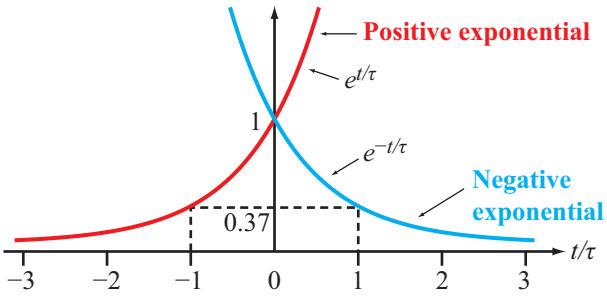



**Figure 1-21:** By $t = \tau$, the exponential function $e^{-t/\tau}$ decays to 37% of its original value at $t = 0$.

▶ An exponential function with a small (short) time constant rises or decays faster than an exponential function with a larger (longer) time constant (**Fig. 1-22(a)**). ◀

Replacing $t$ in the exponential with $(t-T)$ shifts the exponential curve to the right if $T$ is positive and to the left if $T$ is negative (**Fig. 1-22(b)**). Multiplying a negative exponential function by $u(t)$ limits its range to $t > 0$ (**Fig. 1-22(c)**), and by extension, an exponential that starts at $t = T$ and then decays with time constant $\tau$ is given by

$$x(t) = e^{-(t-T)/\tau}\, u(t - T).$$

Its waveform is displayed in **Fig. 1-22(d)**.

Occasionally, we encounter waveforms with the shape shown in **Fig. 1-22(e)**, wherein $x(t)$ starts at zero and builds up as a function of time towards a saturation value. An example is the voltage response of an initially uncharged capacitor,

$$v(t) = V_0(1 - e^{-t/\tau})\, u(t).$$

Table 1-2 provides a general summary of the shapes and expressions of the five nonperiodic waveforms we reviewed in this section.

**Concept Question 1-10:** If the time constant of a negative exponential function is doubled in value, will the corresponding waveform decay faster or slower? (See S²)

**Concept Question 1-11:** What is the approximate shape of the waveform described by the function $(1 - e^{-|t|})$? (See S²)

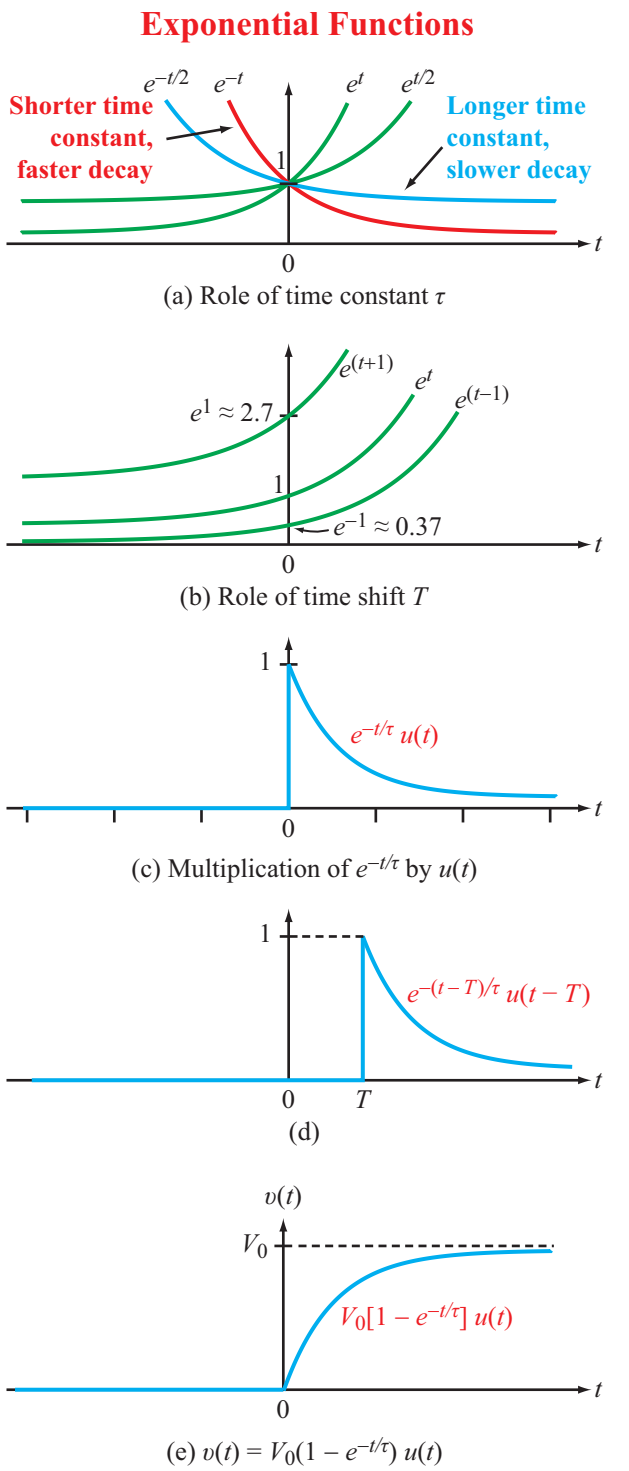
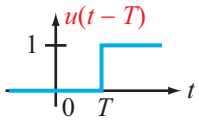
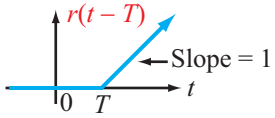
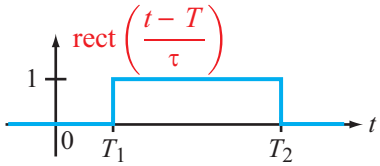
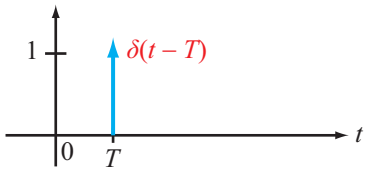
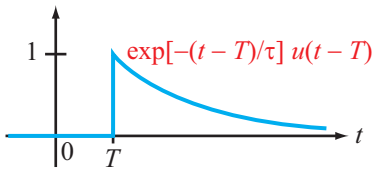



**Figure 1-22:** Properties of the exponential function.

**Table 1-2:** Common nonperiodic functions.

| Function | Expression | General Shape |
|---|---|---|
| Step | $u(t-T)=\begin{cases}0 & \text{for } t<T\\ 1 & \text{for } t>T\end{cases}$ |   |
| Ramp | $r(t-T)=(t-T)\,u(t-T)$ |   |
| Rectangle | $\text{rect}\left(\frac{t-T}{\tau}\right)=u(t-T_1)-u(t-T_2)$ <br> $T_1=T-\frac{\tau}{2}\,;\quad T_2=T+\frac{\tau}{2}$ |   |
| Impulse | $\delta(t-T)$ |   |
| Exponential | $\exp[-(t-T)/\tau]\,u(t-T)$ |   |

**Exercise 1-8:** The radioactive decay equation for a certain material is given by $n(t)=n_0e^{-t/\tau}$, where $n_0$ is the initial count at $t=0$. If $\tau=2\times10^8$ s, how long is its half-life? [Half-life $t_{1/2}$ is the time it takes a material to decay to 50% of its initial value.]

**Answer:** $t_{1/2}=1.386\times10^8$ s $\approx 4$ years. (See S²)

**Exercise 1-9:** If the current $i(t)$ through a resistor $R$ decays exponentially with a time constant $\tau$, what is the ratio of the power dissipated in the resistor at $t=\tau$ to its value at $t=0$?

**Answer:** $p(t)=i^2R=I_0^2R(e^{-t/\tau})^2=I_0^2Re^{-2t/\tau}$, $p(\tau)/p(0)=e^{-2}=0.135$, or 13.5%. (See S²)

## 1-5 Signal Power and Energy

The instantaneous power $p(t)$ dissipated in a resistor $R$ due to the flow of current $i(t)$ through it is

$$p(t)=i^2(t)\,R. \tag{1.31}$$

Additionally, the associated energy expended over a time interval $t_1<t<t_2$ is

$$E=\int_{t_1}^{t_2}p(t)\,dt. \tag{1.32}$$

The expressions for power and energy associated with a resistor can be extended to characterize the ***instantaneous power*** and total ***energy*** of any signal $x(t)$—whether electrical or not and whether real or complex—as

$$p(t)=|x(t)|^2 \tag{1.33a}$$

and

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt, \qquad (1.33b)$$

where $E$ is defined as the ***total energy*** over an infinite time interval $(-\infty < t < \infty)$.

If $|x(t)|^2$ does not approach zero as $t \to \pm\infty$, the integral in Eq. (1.33b) will not converge. In that case, $E$ is infinite, rendering it unsuitable as a measure of the signal's energy capacity. An alternative measure is the ***(time) average power*** $P_{\text{av}}$, which is defined as the power $p(t)$ averaged over all time:

$$P_{\text{av}} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt. \qquad (1.34)$$

Conversely, if $E$ is finite, $P_{\text{av}}$ becomes the unsuitable measure, because in view of Eq. (1.33b), we have

$$P_{\text{av}} = \lim_{T\to\infty} \frac{E}{T} = 0 \qquad (E \text{ finite}). \qquad (1.35)$$

▶ $P_{\text{av}}$ and $E$ define three classes of signals:

**(a)** ***Power signals***: $P_{\text{av}}$ is finite and $E \to \infty$

**(b)** ***Energy signals***: $P_{\text{av}} = 0$ and $E$ is finite

**(c)** ***Non-physical signals***: $P_{\text{av}} \to \infty$ and $E \to \infty$ ◀

For a periodic signal of period $T_0$, it is not necessary to evaluate the integral in Eq. (1.34) with $T \to \infty$; integration over a single period is sufficient. That is,

$$P_{\text{av}} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2\, dt \quad \textbf{(periodic signal)}. \qquad (1.36)$$

Most periodic signals have finite average power; hence, they qualify as power signals.

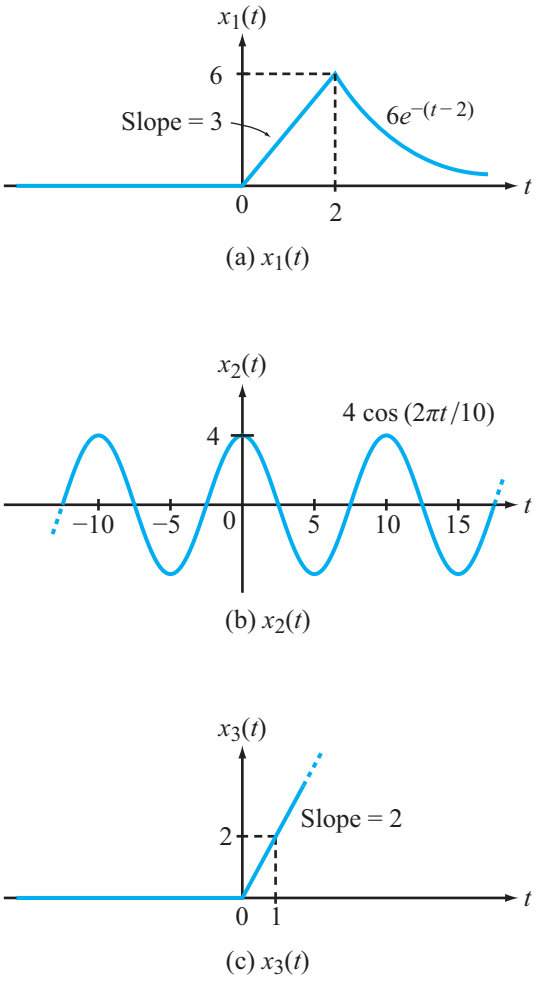



**Figure 1-23:** Signal waveforms for Example 1-7.

**Example 1-7: Power and Energy**

Evaluate $P_{\text{av}}$ and $E$ for each of the three signals displayed in **Fig. 1-23**.

**Solution:**

**(a)** Signal $x_1(t)$ is given by

$$x_1(t) = \begin{cases} 0 & \text{for } t \le 0, \\ 3t & \text{for } 0 \le t \le 2, \\ 6e^{-(t-2)} & \text{for } t \ge 2. \end{cases}$$

Its total energy $E$ is

$$\begin{aligned} E_1 &= \int_0^2 (3t)^2\,dt + \int_2^\infty [6e^{-(t-2)}]^2\,dt \\ &= \int_0^2 9t^2\,dt + \int_2^\infty 36e^{-2(t-2)}\,dt \\ &= \frac{9t^3}{3}\bigg|_0^2 + 36e^4 \int_2^\infty e^{-2t}\,dt \\ &= 24 + 36e^4\left(\frac{-e^{-2t}}{2}\bigg|_2^\infty\right) \\ &= 42. \end{aligned}$$

Note that the second integral represents the energy of $6e^{-t}\,u(t)$ delayed by 2 s. Since delaying a signal does not alter its energy, an alternative method for evaluating the second integral is

$$\begin{aligned} \int_0^\infty (6e^{-t})^2\,dt &= 36\int_0^\infty e^{-2t}\,dt \\ &= 18. \end{aligned}$$

Since $E_1$ is finite, it follows from Eq. (1.35) that $P_{\text{av}_1} = 0$.

**(b)** Signal $x_2(t)$ is a periodic signal given by

$$x_2(t) = 4\cos\left(\frac{2\pi t}{10}\right).$$

From the argument of $\cos(2\pi t/10)$, the period is 10 s. Hence, application of Eq. (1.36) leads to

$$\begin{aligned} P_{\text{av}_2} &= \frac{1}{10}\int_{-5}^{5}\left[4\cos\left(\frac{2\pi t}{10}\right)\right]^2 dt \\ &= \frac{1}{10}\int_{-5}^{5} 16\cos^2\left(\frac{2\pi t}{10}\right) dt \\ &= 8. \end{aligned}$$

The integration was facilitated by the integral relation

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\cos^2\left(\frac{2\pi nt}{T_0}+\phi\right)dt = \frac{1}{2}, \tag{1.37}$$

which is valid for any value of $\phi$ and any integer value of $n$ equal to or greater than 1. In fact, because of Eq. (1.37),

$$P_{\text{av}} = \frac{A^2}{2} \quad \left(\begin{array}{c}\textbf{for any sinusoidal}\\ \textbf{signal of amplitude } A\\ \textbf{and nonzero frequency}\end{array}\right). \tag{1.38}$$

If $\omega_0 = 0$, $P_{\text{av}} = A^2$, not $A^2/2$. Since $P_{\text{av}_2}$ is finite, it follows that $E_2 \to \infty$.

**(c)** Signal $x_3(t)$ is given by

$$x_3(t) = 2r(t) = \begin{cases} 0 & \text{for } t \le 0, \\ 2t & \text{for } t \ge 0. \end{cases}$$

The time-averaged power associated with $x_3(t)$ is

$$\begin{aligned} P_{\text{av}_3} &= \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} 4t^2\,dt \\ &= \lim_{T\to\infty}\frac{1}{T}\left[\frac{4t^3}{3}\bigg|_0^{T/2}\right] \\ &= \lim_{T\to\infty}\left[\frac{1}{T}\times\frac{4T^3}{24}\right] \\ &= \lim_{T\to\infty}\left[\frac{T^2}{6}\right] \to \infty. \end{aligned}$$

Moreover, $E_3 \to \infty$ as well.

**Concept Question 1-12:** Signals are divided into three power/energy classes. What are they? (See S²)

**Exercise 1-10:** Determine the values of $P_{\text{av}}$ and $E$ for a pulse signal given by $x(t) = 5\,\text{rect}\left(\frac{t-3}{4}\right)$.

**Answer:** $P_{\text{av}} = 0$ and $E = 100$. (See S²)

# Summary

## Concepts

- A signal may be continuous, discrete, analog, or digital. It may vary with time, space, or some other independent variable and may be single or multidimensional.
- Signals are classified as causal, noncausal, or anticausal, according to when they start and end.
- Signals can undergo time-shift, time-scaling, and time-reversal transformations.
- A signal may exhibit even or odd symmetry. A signal with neither form of symmetry can be synthesized as the sum of two component signals: one with even symmetry and the other with odd symmetry.
- Real-world signal waveforms often are modeled in terms of a set of elementary waveforms, which include the step, ramp, pulse, impulse, and exponential waveforms.
- A signal's energy capacity is characterized by its average power $P_{av}$ and total energy $E$. These attributes are defined for any signal, whether electrical or not.

## Mathematical and Physical Models

### Signal Transformations

| | |
|---|---|
| Time shift | $y(t) = x(t - T)$ |
| Time scaling | $y(t) = x(at)$ |
| Time reversal | $y(t) = x(-t)$ |

### Signal Symmetry

| | |
|---|---|
| Even | $x(t) = x(-t)$ |
| Odd | $x(t) = -x(-t)$ |
| Even part | $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$ |
| Odd part | $x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$ |
| Sum | $x(t) = x_e(t) + x_o(t)$ |

### Signal Waveforms

See Table 1-2

### Signal Power and Energy

$$P_{av} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

## Important Terms

Provide definitions or explain the meaning of the following terms:

analog signal
anticausal signal
causal signal
continuous signal
digital signal
Dirac delta function
discrete signal
even symmetry
exponential waveform
impulse function
noncausal signal
nonperiodic (aperiodic)
odd symmetry
periodic
physically realizable system
pulse waveform
ramp function
sampling property
signal power
signal energy
singularity
time constant
time reversal
time-scaled
time-shifted
unit rectangular
unit step function

# PROBLEMS

## Section 1-1: Types of Signals

**1.1** Is each of these 1-D signals:

- Analog or digital?
- Continuous-time or discrete-time?

**(a)** Daily closes of the stock market
**(b)** Output from phonograph-record pickup
**(c)** Output from compact-disc pickup

**1.2** Is each of these 2-D signals:

- Analog or digital?
- Continuous-space or discrete-space?

**(a)** Image in a telescope eyepiece
**(b)** Image displayed on digital TV
**(c)** Image stored in a digital camera

**1.3** The following signals are 2-D in space and 1-D in time, so they are 3-D signals. Is each of these 3-D signals:

- Analog or digital?
- Continuous or discrete?

**(a)** The world as you see it
*(b) A movie stored on film
**(c)** A movie stored on a DVD

## Section 1-2: Signal Transformations

**1.4** Given the waveform of $x_1(t)$ shown in **Fig. P1.4(a)**, generate and plot the waveform of:

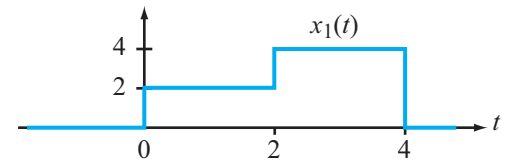
**(a)** $x_1(-2t)$
**(b)** $x_1[-2(t-1)]$

**1.5** Given the waveform of $x_2(t)$ shown in **Fig. P1.4(b)**, generate and plot the waveform of:

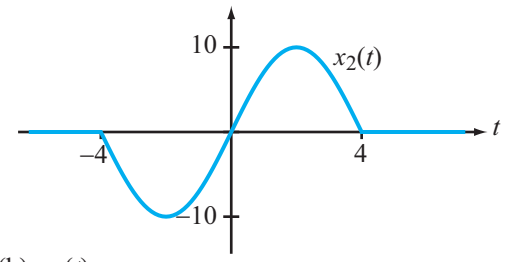
**(a)** $x_2[-(t+2)/2]$
**(b)** $x_2[-(t-2)/2]$

**1.6** Given the waveform of $x_3(t)$ shown in **Fig. P1.4(c)**, generate and plot the waveform of:

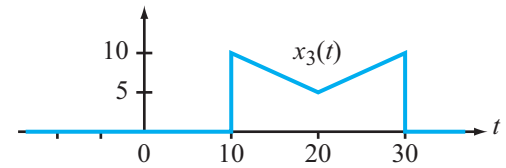
*Answer(s) in Appendix F.



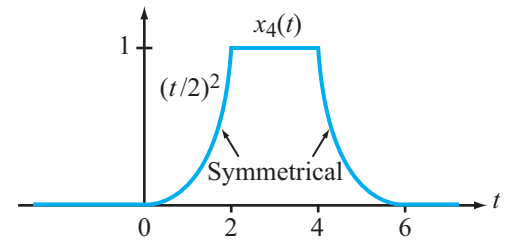
(a) $x_1(t)$



(b) $x_2(t)$



(c) $x_3(t)$



(d) $x_4(t)$

**Figure P1.4:** Waveforms for Problems 1.4 to 1.7.

*(a) $x_3[-(t+40)]$
**(b)** $x_3(-2t)$

**1.7** The waveform shown in **Fig. P1.4(d)** is given by

$$x_4(t) = \begin{cases} 0 & \text{for } t \le 0, \\ \left(\frac{t}{2}\right)^2 & \text{for } 0 \le t \le 2 \text{ s}, \\ 1 & \text{for } 2 \le t \le 4 \text{ s}, \\ f(t) & \text{for } 4 \le t \le 6 \text{ s}, \\ 0 & \text{for } t \ge 6 \text{ s}. \end{cases}$$

**(a)** Obtain an expression for $f(t)$, which is the segment covering the time duration between 4 s and 6 s.

**(b)** Obtain an expression for $x_4(-(t-4))$ and plot it.

**1.8** If

$$x(t) = \begin{cases} 0 & \text{for } t \le 2 \\ (2t-4) & \text{for } t \ge 2, \end{cases}$$

plot $x(t)$, $x(t+1)$, $x\left(\frac{t+1}{2}\right)$, and $x\left(-\frac{(t+1)}{2}\right)$.

**1.9** Given $x(t) = 10(1 - e^{-|t|})$, plot $x(-t+1)$.

**1.10** Given $x(t) = 5\sin^2(6\pi t)$, plot $x(t-3)$ and $x(3-t)$.

**1.11** Given the waveform of $x(t)$ shown in **Fig. P1.11(a)**, generate and plot the waveform of:

**(a)** $x(2t+6)$

*(b) $x(-2t+6)$

**(c)** $x(-2t-6)$

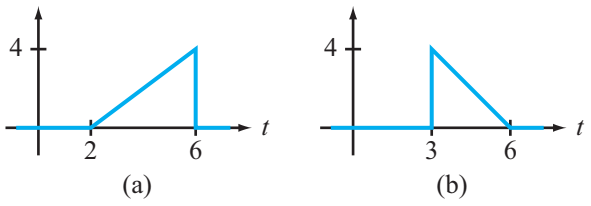



**Figure P1.11:** Waveforms for Problems 1.11 and 1.12.

**1.12** Given the waveform of $x(t)$ shown in **Fig. P1.11(b)**, generate and plot the waveform of:

**(a)** $x(3t+6)$

**(b)** $x(-3t+6)$

**(c)** $x(-3t-6)$

**1.13** If $x(t) = 0$ unless $a \le t \le b$, and $y(t) = x(ct+d)$ unless $e \le t \le f$, compute $e$ and $f$ in terms of $a$, $b$, $c$, and $d$. Assume $c > 0$ to make things easier for you.

**1.14** If $x(t)$ is a musical note signal, what is $y(t) = x(4t)$? Consider sinusoidal $x(t)$.

**1.15** Give an example of a non-constant signal that has the property $x(t) = x(at)$ for all $a > 0$.

## Sections 1-3 and 1-4: Waveforms

**1.16** For each of the following functions, indicate if it exhibits even symmetry, odd symmetry, or neither one.

**(a)** $x_1(t) = 3t^2 + 4t^4$

*(b) b $x_2(t) = 3t^3$

**1.17** For each of the following functions, indicate if it exhibits even symmetry, odd symmetry, or neither one.

**(a)** $x_1(t) = 4[\sin(3t) + \cos(3t)]$

**(b)** $x_2(t) = \dfrac{\sin(4t)}{4t}$

**1.18** For each of the following functions, indicate if it exhibits even symmetry, odd symmetry, or neither one.

**(a)** $x_1(t) = 1 - e^{-2t}$

**(b)** $x_2(t) = 1 - e^{-2t^2}$

**1.19** Generate plots for each of the following step-function waveforms over the time span from $-5$ s to $+5$ s.

**(a)** $x_1(t) = -6u(t+3)$

**(b)** $x_2(t) = 10u(t-4)$

**(c)** $x_3(t) = 4u(t+2) - 4u(t-2)$

**1.20** Generate plots for each of the following step-function waveforms over the time span from $-5$ s to $+5$ s.

**(a)** $x_1(t) = 8u(t-2) + 2u(t-4)$

*(b) $x_2(t) = 8u(t-2) - 2u(t-4)$

**(c)** $x_3(t) = -2u(t+2) + 2u(t+4)$

**1.21** Provide expressions in terms of step functions for the waveforms displayed in **Fig. P1.21**.

**1.22** Generate plots for each of the following functions over the time span from $-4$ s to $+4$ s.

**(a)** $x_1(t) = 5r(t+2) - 5r(t)$

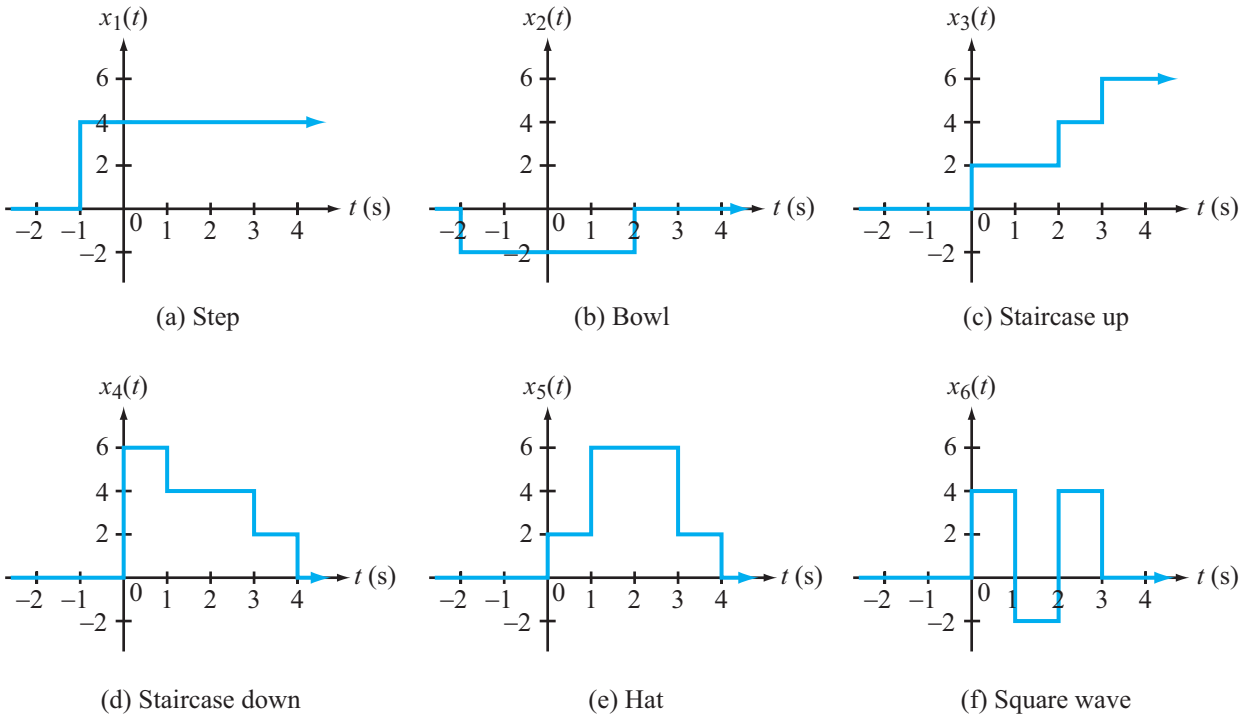
**(b)** $x_2(t) = 5r(t+2) - 5r(t) - 10u(t)$

*(c) $x_3(t) = 10 - 5r(t+2) + 5r(t)$

**(d)** $x_4(t) = 10\text{rect}\left(\dfrac{t+1}{2}\right) - 10\text{rect}\left(\dfrac{t-3}{2}\right)$

**(e)** $x_5(t) = 5\text{rect}\left(\dfrac{t-1}{2}\right) - 5\text{rect}\left(\dfrac{t-3}{2}\right)$

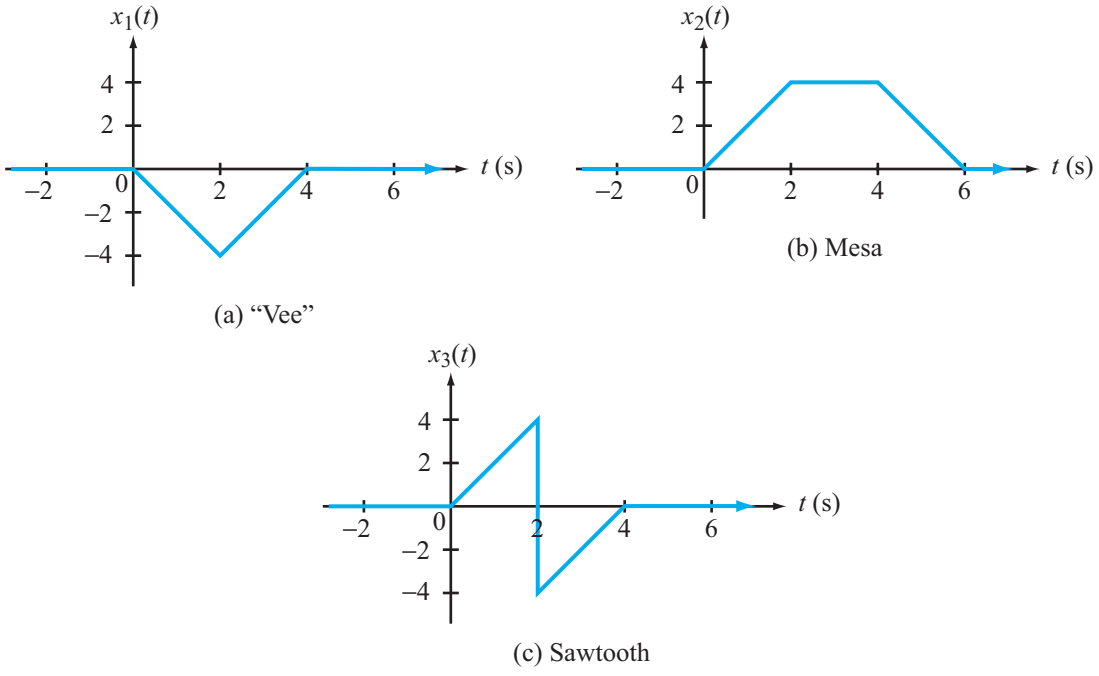
**1.23** Provide expressions for the waveforms displayed in **Fig. P1.23** in terms of ramp and step functions.




(a) Step (b) Bowl (c) Staircase up

(d) Staircase down (e) Hat (f) Square wave

**Figure P1.21:** Waveforms for Problem 1.21.




(a) "Vee" (b) Mesa

(c) Sawtooth

**Figure P1.23:** Waveforms for Problem 1.23.

**1.24** For each of the following functions, indicate if its waveform exhibits even symmetry, odd symmetry, or neither.

**(a)** $x_1(t) = u(t-3) + u(-t-3)$

**(b)** $x_2(t) = \sin(2t)\cos(2t)$

**(c)** $x_3(t) = \sin(t^2)$

**1.25** Provide plots for the following functions over a time span and with a time scale that will appropriately display the shape of the associated waveform of:

**(a)** $x_1(t) = 100e^{-2t}\,u(t)$

**(b)** $x_2(t) = -10e^{-0.1t}\,u(t)$

**(c)** $x_3(t) = -10e^{-0.1t}\,u(t-5)$

**(d)** $x_4(t) = 10(1-e^{-10^3 t})\,u(t)$

**(e)** $x_5(t) = 10e^{-0.2(t-4)}\,u(t)$

**(f)** $x_6(t) = 10e^{-0.2(t-4)}\,u(t-4)$

**1.26** Determine the period of each of the following waveforms.

**(a)** $x_1(t) = \sin 2t$

**(b)** $x_2(t) = \cos\left(\frac{\pi}{3}\,t\right)$

**(c)** $x_3(t) = \cos^2\left(\frac{\pi}{3}\,t\right)$

*(d) $x_4(t) = \cos(4\pi t + 60°) - \sin(4\pi t + 60°)$

**(e)** $x_5(t) = \cos\left(\frac{4}{\pi}\,t + 30°\right) - \sin(4\pi t + 30°)$

**1.27** Provide expressions for the waveforms displayed in **Fig. P1.27** in terms of ramp and step functions.

**1.28** Use the sampling property of impulses to compute the following.

**(a)** $y_1(t) = \int_{-\infty}^{\infty} t^3\,\delta(t-2)\,dt$

**(b)** $y_2(t) = \int_{-\infty}^{\infty} \cos(t)\,\delta(t-\pi/3)\,dt$

**(c)** $y_3(t) = \int_{-3}^{-1} t^5\,\delta(t+2)\,dt$

**1.29** Use the sampling property of impulses to compute the following.

**(a)** $y_1(t) = \int_{-\infty}^{\infty} t^3\,\delta(3t-6)\,dt$

*(b) $y_2(t) = \int_{-\infty}^{\infty} \cos(t)\,\delta(3t-\pi)\,dt$

**(c)** $y_3(t) = \int_{-3}^{-1} t^5\,\delta(3t+2)\,dt$

**1.30** Determine the period of each of the following waveforms.

**(a)** $x_1(t) = 6\cos\left(\frac{2\pi}{3}\,t\right) + 7\cos\left(\frac{\pi}{2}\,t\right)$

**(b)** $x_2(t) = 6\cos\left(\frac{2\pi}{3}\,t\right) + 7\cos(\pi\sqrt{2}\,t)$

**(c)** $x_3(t) = 6\cos\left(\frac{2\pi}{3}\,t\right) + 7\cos\left(\frac{2}{3}\,t\right)$

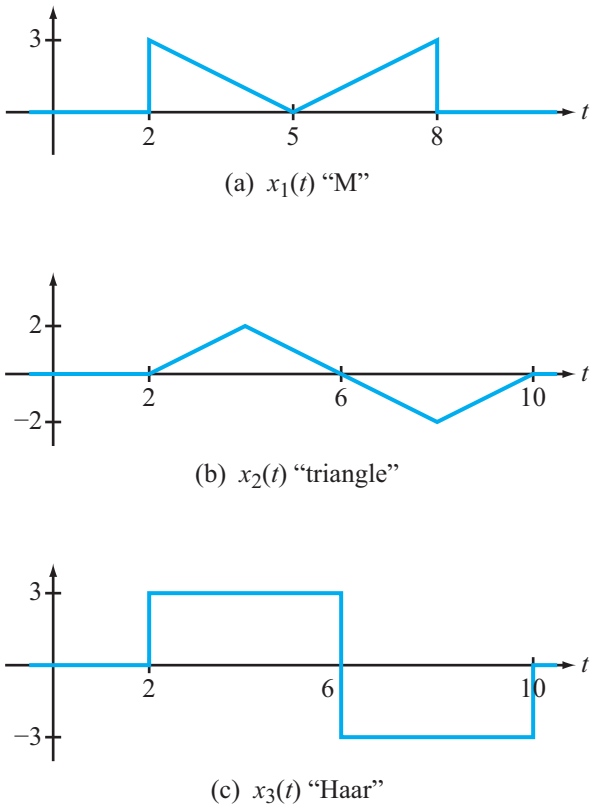



**Figure P1.27:** Waveforms for Problem 1.27.

**1.31** Determine the period of each of the following functions.

**(a)** $x_1(t) = (3+j2)e^{j\pi t/3}$

**(b)** $x_2(t) = (1+j2)e^{j2\pi t/3} + (4+j5)e^{j2\pi t/6}$

**(c)** $x_3(t) = (1+j2)e^{jt/3} + (4+j5)e^{jt/2}$

**1.32** If $M$ and $N$ are both positive integers, provide a general expression for the period of

$$A\cos\left(\frac{2\pi}{M}\,t+\theta\right) + B\cos\left(\frac{2\pi}{N}\,t+\phi\right).$$

## Sections 1-5: Power and Energy

**1.33** Determine if each of the following signals is a power signal, an energy signal, or neither.

**(a)** $x_1(t) = 3[u(t+2) - u(t-2)]$

**(b)** $x_2(t) = 3[u(t-2) - u(t+2)]$

**(c)** $x_3(t) = 2[r(t) - r(t-2)]$

**(d)** $x_4(t) = e^{-2t}\,u(t)$

**1.34** Determine if each of the following signals is a power signal, an energy signal, or neither.

**(a)** $x_1(t) = [1 - e^{2t}]\, u(t)$

*__(b)__ $x_2(t) = [t \cos(3t)]\, u(t)$

**(c)** $x_3(t) = [e^{-2t} \sin(t)]\, u(t)$

**1.35** Determine if each of the following signals is a power signal, an energy signal, or neither.

**(a)** $x_1(t) = [1 - e^{-2t}]\, u(t)$

**(b)** $x_2(t) = 2 \sin(4t) \cos(4t)$

**(c)** $x_3(t) = 2 \sin(3t) \cos(4t)$

**1.36** Use the notation for a signal $x(t)$:

$E[x(t)]$: total energy of the signal $x(t)$,
$P_{\text{av}}[x(t)]$: average power of the signal $x(t)$ if $x(t)$ is periodic.

Prove each of the following energy and power properties.

**(a)**

$$E[x(t+b)] = E[x(t)]$$

and

$$P_{\text{av}}[x(t+b)] = P_{\text{av}}[x(t)]$$

(time shifts do not affect power or energy).

**(b)**

$$E[ax(t)] = |a|^2\, E[x(t)]$$

and

$$P_{\text{av}}[ax(t)] = |a|^2\, P_{\text{av}}[x(t)]$$

(scaling by $a$ scales energy and power by $|a|^2$).

**(c)**

$$E[x(at)] = \frac{1}{a}\, E[x(t)]$$

and

$$P_{\text{av}}[x(at)] = P_{\text{av}}[x(t)]$$

if $a > 0$ (time scaling scales energy by $\frac{1}{a}$ but doesn't affect power).

**1.37** Use the properties of Problem 1.36 to compute the energy of the three signals in **Fig. P1.27**.

**1.38** Compute the energy of the following signals.

**(a)** $x_1(t) = e^{-at}\, u(t)$ for $a > 0$

**(b)** $x_2(t) = e^{-a|t|}$ for $a > 0$

**(c)** $x_3(t) = (1 - |t|)\, \text{rect}(t/2)$

**1.39** Compute the average power of the following signals.

**(a)** $x_1(t) = e^{jat}$ for real-valued $a$

**(b)** $x_2(t) = (3 - j4)e^{j7t}$

*__(c)__ c $x_3(t) = e^{j3}e^{j5t}$

**1.40** Prove these energy properties.

**(a)** If the even-odd decomposition of $x(t)$ is

$$x(t) = x_{\text{e}}(t) + x_{\text{o}}(t),$$

then

$$E[x(t)] = E[x_{\text{e}}(t)] + E[x_{\text{o}}(t)].$$

**(b)** If the causal-anticausal decomposition of $x(t)$ is $x(t) = x(t)\, u(t) + x(t)\, u(-t)$, then

$$E[x(t)] = E[x(t)\, u(t)] + E[x(t)\, u(-t)].$$